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A Systematic Extended Iterative Solution for Quantum Chromodynamics

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Abstract: An outline is given of an extended perturbative solution of Euclidean QCD which systematically accounts for a class of nonperturbative effects, while still allowing renormalization by the perturbative counterterms. Euclidean proper vertices Γ are approximated by a double sequence $\Gamma^{[r,p]}$, where r denotes the degree of rational approximation with respect to the spontaneous mass scale Λ_{QCD}, nonanalytic in the coupling g^2, while p represents the order of perturbative corrections in g^2 calculated from $\Gamma^{[r,0]}$ – rather than from the perturbative Feynman rules $\Gamma^{(0)\text{pert}}$ – as a starting point. The mechanism allowing the nonperturbative terms to reproduce themselves in the Dyson-Schwinger equations preserves perturbative renormalizability and is intimately tied to the divergence structure of the theory. As a result, it restricts the self-consistency problem for the $\Gamma^{[r,0]}$ rigorously – i.e. without decoupling approximations – to the seven superficially divergent vertices. An interesting aspect of the solution is that rational-function sequences for the QCD propagators contain subsequences describing short-lived elementary excitations. The method is calculational, in that it allows the known techniques of loop computation to be used while dealing with integrands of truly nonperturbative content.

1 Generalities and Notation

1.1 Nonperturbative Quantities

One of the more important insights to have emerged from two decades of study of the large-order behavior and summability of perturbation expansions [1] has been that for renormalizable but not superrenormalizable field theories (typically in four dimensions), the perturbative series of correlation functions $\Gamma(g^2)$ around $g^2 = 0$ is fundamentally incomplete, in the sense that it does not allow unique reconstruction of those functions even in principle. Perturbation series, in most cases of interest, are divergent asymptotic series which at fixed positive g^2 allow a function to be estimated with a finite accuracy. A typical error estimate is

$$\min_{\{p\}} \left| \Gamma(g^2) - \sum_{p'=0}^{p} \Gamma^{(p')\mathrm{pert}} \left(\frac{g}{4\pi} \right)^{2p'} \right| \propto e^{-\frac{\mathrm{const.}}{g^2}} , \tag{1.1}$$

as can be inferred from the behavior of the perturbative coefficients $\Gamma^{(p)\mathrm{pert}}$ generally found in quantum field theories [1]. For superrenormalizable theories, typically in dimensions $D \leq 3$, analyticity properties of the $\Gamma's$ in the complex g^2 plane can be established outside of perturbation theory that are strong enough to conclude that a function sufficiently analytic and satisfying this bound must in fact vanish, and that $\Gamma's$ can be reconstructed uniquely from their asymptotic series by resummation techniques. By contrast, for renormalizable theories in $D = 4$, the necessary analyticity is positively known to be violated [2], and $\Gamma's$ may therefore differ from their perturbative series by terms exponentially small near $g^2 = 0$ that can "duck under" the bound (1.1) and remain invisible in the expansion.

The general mathematics of semi-convergent series does not say more about the missing terms, if any, that have no expansion in g^2 in even a formal sense : one must turn to physics for clues. Asymptotically free theories are special in that there one has a-priori knowledge, from renormalization-group (RG) analysis, of the presence of one important class of terms just allowed by the bound (1.1). These are the terms involving the RG-invariant spontaneous mass scale [3],

$$\Lambda^2 \left(g^2(\nu), \nu \right)_R = \nu^2 \exp \left\{ -2 \int^{g(\nu)} dg' \, \frac{1}{[\beta(g')]_R} \right\} , \tag{1.2}$$

where ν is the arbitrary mass scale introduced by renormalization in a scheme R. With asymptotic freedom,

$$\Lambda^2 = \nu^2 \exp \left\{ - \frac{1}{\beta_0 [g(\nu)/4\pi]^2} [1 + O(g^2)] \right\} , \tag{1.3}$$

(where β_0, defined by eq. (1.20) below, is positive), the archetypal non-analytic g^2 dependence of the "exponentially small" form. It hardly needs emphasis that the term "exponentially small" refers to

the hypothetical behavior of such a quantity in a purely formal, unphysical limit – the limit where one sends $g^2 \to 0$ *without* a corresponding increase in ν in the sense of RG flow. In the commonly used (dimensional) schemes R, Λ_R of course is not small, and in fact sets the scale of hadronic masses.

It is also clear from the context that when speaking of Λ^2-dependent terms we are *not* alluding here to the relatively weak Λ^2 dependence obtained by the standard RG process of introducing a running coupling constant $\bar{\alpha}(k^2/\Lambda^2)$, with asymptotic expansion

$$\frac{\bar{\alpha}(z)}{4\pi} = \frac{1}{\beta_0 \ln z}\left\{1 + O\left(\frac{1}{\ln z}\right)\right\} \qquad \left(z = \frac{k^2}{\Lambda^2} >> 1\right), \tag{1.4}$$

since this arises purely *within* the perturbation series by leading-logarithms (LL) resummation. By contrast, consider some quantities presumed to be nonperturbative in the deeper sense mentioned above, such as the vacuum expectation of the trace anomaly [4] of the energy-momentum tensor in massless QCD. Since this is known to be an RG invariant and of mass dimension four, it must be of the form

$$< 0 \left| \frac{\beta(g)}{2g} G_a^{\mu\nu}\, G_a^{\mu\nu} \right| 0 >_R = c_A \Lambda^4\,, \tag{1.5}$$

where c_A is a pure number. As another example, again in massless QCD, consider dynamical formation of a color-singlet glueball state, whose mass defines an RG-invariant scale. The connected and amputated four-gluon amplitude takes the form,

$$T^{\kappa\lambda\mu\nu}_{abcd}(p_1 \dots p_4) = \delta_{ab}\delta_{cd}\frac{\Phi^{\kappa\lambda}(p_1,p_2)\Phi^{\mu\nu}(p_3,p_4)}{(p_1+p_2)^2 + c_B\Lambda^2} \tag{1.6}$$

$$+(\text{crossed terms}) + (\text{regular terms})\,,$$

with c_B again a pure number. Yet another example is furnished by instanton effects. There, n-instanton contributions to physical quantities typically appear with factors $\exp(-nS_1/g^2)$, where the instanton action S_1 is in general *not* an integer multiple of the $(4\pi)^2/\beta_0$ of eq. (1.3), so that fractional powers of Λ seem to be present. However, strong arguments have been given by Münster [5] to the effect that this is really an artefact of the dilute-gas approximation used in most instanton calculations, and that by taking account of the denseness of the instanton system one is led back to integer powers of Λ in the properly summed multi-instanton series for physical quantities. These examples indicate, as does experience with operator-product expansions, that the Λ dependence of truly nonperturbative quantities will typically be of a *polynomial or rational* form. It is this type of dependence which can be clearly separated, without double-counting problems, from the "resummed-perturbative" one of (1.4), and for which *g^2 and Λ^2 can be handled formally like two different parameters,* (although in the end one will always seek to use LL resummation to eliminate g^2 and deal only with a single, RG-invariant parameter, Λ^2).

The present paper outlines [6], for an Euclidean and asymptotically free gauge theory exemplified by QCD, a systematic extension of perturbation theory designed to account for *nonperturbative effects as reflected in a rational dependence, or more generally a dependence representable by sequences of rational functions,* of correlation functions on Λ. Since the approximating sequence is to provide a genuine extension, rather than a resummation, of the perturbative one, it is natural that it should take the form of a *double sequence*, $\Gamma^{[r,p]}$, as described in sect. 2 for the simplest case of a one-component vertex function depending on a single squared Euclidean momentum (the two-point vertex of transverse gluons); sect. 3 and appendix A describe the relatively straightforward extension to the more complex (three- and four-point) basic vertex functions. In this double sequence, the integer r will characterize a certain degree of approximation with respect to Λ, while the "perturbative" index p still counts explicit powers of $g^2(\nu)$. That r cannot simply be an index counting powers of Λ has to do with the fundamental difference of dimensionality: with Λ being a mass, as opposed to the dimensionless g^2, an expansion in powers of Λ would (for fixed mass dimension d_Γ of Γ) inevitably be a large-Q^2 expansion in powers of Λ^2/Q^2, where Q^2 is a typical squared Euclidean four-momentum of Γ, and such an expansion would never provide an adequate representation of Γ in the region $Q^2 \leq \Lambda^2$, whereas the dynamical equations determining Γ's contain loop integrals whose evaluation requires a systematic approximation of those Γ's over the entire momentum range. Of necessity, the index r will therefore have to refer to a sequence of more *global* approximants, whereas for g^2, which enters the dynamical equations only parametrically, a local approximation around $g^2 = 0$ is possible. The peculiar asymmetry between the two "directions" of the sequence is therefore not an arbitrary choice, but is rooted in the very nature of the spontaneous scale (1.2).

On the other side, the meaning of p will also be subtly different from what it is in the purely perturbative context, since for $p \geq 1$ it will refer to corrections calculated from diagrammatic building blocks $\Gamma^{[r,0]}$, rather than from the standard Feynman-rules vertices $\Gamma^{(0)pert}$ implied in eq. (1.1). In other words, as compared to perturbation theory, the approximation will be able to "correct its own zeroth order". Determination of the set $\{\Gamma^{[r,0]}\}$, the *nonperturbatively modified vertices of zeroth perturbative order*, will then lead to the self-consistency problem outlined in sect. 4, which is one of the characteristic new features of the scheme. We discuss the remarkable fact that this self-consistency problem becomes *rigorously* restricted, not by any "decoupling" approximations but by the very nature of a mechanism tied to the divergence structure of the theory, to a small finite set of vertices, and we indicate how perturbative renormalization continues to function in the framework of this mechanism. Finally, section 5 has a number of comments on the special description emerging in this context of the elementary QCD excitations. Apart from a few glimpses provided by the sample calculation of sect. 4, this outline does not dwell at all on either the bulk of the actual loop and self-consistency computations, nor on the interesting, but nevertheless dynamically secondary, question of

approximate nonperturbative saturation of the Slavnov-Taylor identities, since both subjects are of an algebraic lengthiness that requires separate presentation. We do, however, comment occasionally on the relation with the work of refs. [7], which the present scheme allows to put in perspective as a lowest stage (although with somewhat arbitrary technical simplifications) of a more systematic sequential approximation.

1.2 Notation and Conventions

The following notation will be used. One considers correlation functions generated by the standard gauge-theory action in $D = 4 - 2\epsilon$ Euclidean dimensions,

$$S = \int d^D x \left[\mathcal{L}_V(x) + \sum_F \mathcal{L}_F(x) + \mathcal{L}_G(x) \right] \tag{1.7}$$

where $\mathcal{L}_V$ and $\mathcal{L}_F$ are Lagrange densities, respectively, of a set of $SU(N_C)$ gauge-vector (gluon) fields and of a set of minimally coupled fermion (quark) fields coming in N_F flavors $F = u, d, s \ldots$, while the term

$$\mathcal{L}_G = \frac{1}{2\xi_0} \left[\partial^\mu A_a^\mu(x)\right]^2 + \bar{c}_a(x) \left\{ \left[-\delta_{ab}\partial^\mu + \tilde{g}_0 f_{abc} A_c^\mu(x)\right] \partial^\mu \right\} c_b(x)\,. \tag{1.8}$$

comprises standard covariant gauge-fixing and Fadde'ev-Popov (FP) terms. The coupling factor $\tilde{g}_0 = \nu_0^\epsilon g_0$, with ν_0 another mass scale, is used to keep the bare gauge coupling g_0 dimensionless. We will focus in particular on the connected, amputated and one-particle irreducible correlation functions (*proper vertices*) with N external lines in Euclidean momentum space,

$$\Gamma_N(\{k\}), \qquad \{k\} = \{k_1, k_2, \ldots k_N | k_1 + k_2 + \ldots + k_N = 0\}\,. \tag{1.9}$$

Where necessary, the N external lines will be labelled more specifically by V for a vector (gluon) line – occasionally detailed further as T or L for a transverse or longitudinal gluon, respectively –, by $G, \bar{G}$ for ghost and antighost lines, and by $F, \bar{F}$ for quark and antiquark lines. Thus Γ_{TLV} will be a purely gluonic three-point vertex with one transverse, one longitudinal, and one generic gluon line, $\Gamma_{FVV\bar{F}}$ a quark-gluon-gluon-antiquark four-point vertex, etc. The numbers of the three types of lines will be written n_V, n_G, n_F, so that

$$n_V + n_G + \sum_F n_F = N\,. \tag{1.10}$$

In particular, the set of proper two-point vertices

$$\Gamma_2 := \left\{ \; \Gamma_{VV} = -D^{-1}, \;\; \Gamma_{G\bar{G}} = -\tilde{D}^{-1}, \;\; \left\{ \Gamma_{F\bar{F}} = -S_F^{-1} | F = 1 \ldots N_F \right\} \; \right\}, \tag{1.11}$$

consists of the negative inverses of the vector propagator D, ghost propagator $\tilde{D}$, and fermion propagators S_F, while the set

$$\Gamma_3 := \{ \; \Gamma_{3V}, \; \Gamma_{GV\bar{G}}, \;\; \{\Gamma_{FV\bar{F}} | F = 1 \ldots N_F\} \; \} \tag{1.12}$$

comprises the basic three-point interaction vertices. The dynamics implied by the action (1.7 – 1.8) are then embodied in the Dyson-Schwinger (DS) equations [8], an infinite, hierarchical system of coupled integral equations for the functions (1.9) (see [9] for their specific form in QCD), which in a condensed notation take the form

$$\Gamma_N = \Gamma_N^{(0)\mathrm{pert}} + \left(\frac{g_0}{4\pi}\right)^2 \Phi_N \left[\Gamma_2, \Gamma_3, \ldots, \Gamma_N, \Gamma_{N+1}, \Gamma_{N+2}\right] . \tag{1.13}$$

The *perturbative zeroth-order or bare vertices*, $\Gamma_N^{(0)\mathrm{pert}}$, are given by the standard Feynman rules for the action (1.7 - 1.8), while Φ_N denotes a set of nonlinear dressing functionals, defined by loop integrals over combinations of $\Gamma's$, and the notation emphasizes that each such loop integral is preceded by at least one power of the bare gauge coupling, g_0^2.

It is important to keep in mind that a fundamental dichotomy is introduced into the set of vertices (1.9) through the renormalizable divergence structure of a QCD-like theory. A small finite subset, consisting of the *superficially divergent or basic vertices*,

$$\Gamma_{sdiv} := \{\ \Gamma_2,\ \Gamma_3,\ \Gamma_{4V}\ \}\ , \tag{1.14}$$

are distinguished by the fact that loop integrals in their Φ_N functionals of eqs. (1.13) have a non-negative value of the overall degree of divergence in $D = 4$ [10],

$$\delta_N = 4 - n_V - \frac{3}{2}\left(n_G + \sum_F n_F\right)\ , \tag{1.15}$$

so that each of these need their own specific renormalizations, which can be performed at least perturbatively [11]. Related to this is the fact that it is precisely for Γ_{sdiv} that the bare terms $\Gamma^{(0)\mathrm{pert}}$ in eq. (1.13) are nonzero. By contrast, the remaining, infinite set of *superficially convergent or higher vertices*,

$$\Gamma_{sconv} := \{\ \Gamma_{GVV\bar{G}},\ \Gamma_{FVV\bar{F}},\ \Gamma_{FF\bar{F}\bar{F}},\ \Gamma_{5V},\ \ldots\} \tag{1.16}$$

have loop integrals with a negative δ_N, and therefore exhibit no typical divergences of their own, but at most subdivergences representing corrections to the basic vertices (1.14): when rewritten as *dressed-skeleton expansions*, i.e. in terms of fully dressed and renormalized basic vertices (1.14), their loops are actually convergent.

While testable consequences of the theory are mostly contained in (the color-singlet channels of) the higher Green's functions (1.16), these cannot be calculated in a truly systematic way without first studying, and renormalizing, the basic vertices (1.14): a disproportionate amount of theoretical effort must be directed towards a class of amplitudes that contain next to nothing in observable physics. (An exception may be the four-gluon vertex Γ_{4V} in (1.14), which may develop glueball poles.) The approach described below is, at its present stage, concerned exclusively with the DS equations (1.13) for the seven superficially divergent vertices (1.14).

The usual *perturbative solution* to eqs. (1.13) is obtained by straightforward iteration around $\Gamma_N^{(0)\mathrm{pert}}$ and by applying, at each step, a renormalization scheme R, which among other things eliminates g_0 in favor of a renormalized coupling $g(\nu)$ depending on the arbitrary scale ν :

$$\Gamma_N^{\mathrm{pert}} = \lim_{p\to\infty} \Gamma_N^{[p]\mathrm{pert}}; \qquad \Gamma_N^{[p]\mathrm{pert}} = \Gamma_N^{(0)\mathrm{pert}} + \sum_{p'=1}^{p} \left[\frac{g(\nu)}{4\pi}\right]^{2p'} \Gamma_N^{(p')\mathrm{pert}} . \tag{1.17}$$

Here the radiative corrections $\Gamma_N^{(p)\mathrm{pert}}$ ($p \geq 1$) are computed iteratively from the zeroth-order solution $\Gamma_N^{(0)\mathrm{pert}}$, i. e. from the standard Feynman rules. For example, the first iteration is described schematically by

$$\left\{ \left(\frac{g_0}{4\pi}\right)^2 \Phi_N \left[\Gamma^{(0)\mathrm{pert}}\right] \right\}_{R,\nu} = \left[\frac{g(\nu)}{4\pi}\right]^2 \Gamma_N^{(1)\mathrm{pert}} + 0(g^4) . \tag{1.18}$$

For R, we always have in mind a dimensional-regularization-plus-minimal-subtraction scheme with respect to $D = 4 - 2\epsilon$, which in particular entails the familiar coupling renormalization

$$g_0^2 \nu_0^{2\epsilon} = g^2(\nu) \nu^{2\epsilon} Z_\alpha(g^2(\nu), \epsilon) , \tag{1.19}$$

where at the one-loop level

$$Z_\alpha = 1 - \beta_0 \left[\frac{g(\nu)}{4\pi}\right]^2 \frac{1}{\epsilon} + 0(g^4) ; \qquad \beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_F . \tag{1.20}$$

1.3 Problems not Addressed

It may be clarifying to mention two related problems which, while important in their own right, are *not* addressed in this paper.

(i) It is by no means clear at this time whether (1.7) with (1.8) is the correct action to use in continuum QCD. Recent sharpening by Zwanziger [12,13] of Gribov's criticism [14] of the insufficient gauge fixing afforded by (1.8) has rendered untenable the convenient prejudice that "Gribov's problem may be ignored in perturbative treatments". It is now quite clear that if one insists on quantizing the chromodynamic system with a non-redundant set of degrees of freedom for the gauge field, the ensuing restriction of the path integral to a fundamental domain, containing exactly one representative of each gauge orbit, affects also the "perturbative" region of small fields. Zwanziger has in fact presented [12] a method of approximately "exponentiating" the fundamental-domain restriction, which demonstrates the minimum of new effects to be expected from a more complete gauge fixing. His treatment leads to the replacement of the action (1.7) by $S' = S + \gamma H$, where γ is a parameter of dimension mass^4 and of the non-analytic type (1.3), while H is a nonlocal "horizon" functional,

$$H = \int d^D x d^D y A_a^\mu(x) f_{cae} \left[M^{-1}(x,y)\right]_{cd} f_{dbe} A_b^\mu(y) , \tag{1.21}$$

with M^{-1} denoting the inverse of the FP operator in the curly bracket of (1.8). (This nonlocal term may be replaced with a local one by using path integration over additional auxiliary fields.) The drastic low-energy effects produced by this term – in particular, the emergence of the gluon propagator of eq. (2.17) below, describing a short-lived elementary excitation of the gluon field – bear several intriguing similarities to those emerging from the "extended perturbative" approach discussed here, but the precise relationship of the two methods is unknown. What should be emphasized at this time is that there is not necessarily a contradiction between them. The Dyson-Schwinger solution for the "redundantly" described system of eqs. (1.7/1.8), when given enough nonperturbative freedom, may settle down self-consistently in the region singled out by the non-redundant description, and the recent insight that the effect of (1.21) can be described in terms of a nonperturbative vacuum spontaneously breaking BRS invariance [15] may be seen as pointing in this direction.

(ii) The present paper studies a purely Euclidean theory. In the few instances where we need to refer to its properties in the Minkowskian domain, as in sect. 5 below, we shall proceed, like most current investigations of QCD, on the assumption that the usual strategy of Euclidean field theory – to define Green's functions and solve for them entirely in the Euclidean, and to continue to the Minkowskian only in the final answers – yields physically correct results. For the gauge-fixing dependent correlations of the *elementary* QCD fields, from which any analytic investigation has to start, there seems to be at present no full proof of this, and the discussion below will show that in the approximating subsequence of primary interest for QCD, the Euclidean solution continued analytically will differ from the corresponding direct solution to the Minkowskian equations. Since the core parts of the method described here, and in particular the basic quantum effect leading to self-consistency of the nonperturbative terms, are general enough to continue to work in e.g. a purely Minkowskian theory, we have not given this question any priority, but its existence should be kept in mind when proceeding to various applications that the method will invite.

2 The Extended Perturbative Expansion

2.1 General Restrictions

We next describe the form and general properties of the extended iterative sequence, with no attention as yet to the question of how this form achieves self-consistency in the DS equations of the theory. In this sequence, as the terminology indicates, the organizing principle of the perturbative solution (1.17) is not discarded altogether: one still considers a formal power series in the parameter $[g(\nu)/4\pi]^2$, and therefore a weak-coupling solution that is directly applicable (i.e., applicable without infinite resummations) only if that parameter remains sufficiently small *at all scales* ν to permit semi-convergent expansions of the Γ_N. It is worth emphasizing that this assumption is entirely compatible with present

fragmentary knowledge about the flow of g in QCD: what is truly known of $g(\nu)$ are a few leading orders of its asymptotic expansion at large ν, which are of the inverse-logarithmic form of eq. (1.4), with k^2 replaced by ν^2. As for low-ν behavior, although folklore about "the running coupling blowing up at the scale Λ" has become so pervasive as to be occasionally confused with theory, the emerging consensus, if any, from phenomenology [16] and lattice studies [17] seems rather to point in the direction anticipated by Gribov in 1987 [18]: as $\nu \to 0$, the running coupling does nothing dramatic, but "freezes" around an order of magnitude of perhaps 0.2 for the quantity $[g(\nu)/2\pi]^2 = \alpha_s/\pi$ in the $\overline{\mathrm{MS}}$ scheme. In the present context, as we shall see, *amplitudes* can get large at momenta $\approx \Lambda$, but via a rather different route opened up by the novel feature of the expansion. The new feature is that we now have a double, two-index sequence of approximants, in which each term is allowed an additional dependence on the g^2-nonanalytic mass parameter (1.3):

$$\Gamma_N(\{k\}; g^2(\nu); \nu) = \lim_{r\to\infty} \lim_{p\to\infty} \Gamma_N^{[r,p]}(\{k\}; g^2(\nu), \nu)\,, \tag{2.1}$$

$$\Gamma_N^{[r,p]} = \Gamma_N^{[r,0]}(\{k\}; \Lambda) + \sum_{p'=1}^{p} \left[\frac{g(\nu)}{4\pi}\right]^{2p'} \Gamma_N^{[r,p')}(\{k\}; \Lambda; \nu) \tag{2.2}$$

(We are using an abbreviated notation suppressing all dependence on parameters not immediately relevant to the present argument, such as quark masses and gauge-fixing parameters.) One may immediately state two *boundary conditions* on the nonperturbatively modified $\Gamma^{[r,p]}$ amplitudes. Since Λ^2, by (1.3), vanishes faster as $g^2 \to 0$ than any power of g^2, it makes physical sense to consider the (formal) limit in which $\Lambda^2 \to 0$ but the g^{2p} remain finite. In this *perturbative limit* we should ensure

$$\Gamma^{[r,p)}(\Lambda^2 = 0) = \Gamma^{(p)\mathrm{pert}} \qquad (p = 0, 1, 2\ldots)\,. \tag{2.3}$$

On the other hand, since QCD is asymptotically free, and since the logarithmic corrections to asymptotic freedom are known to arise from partial resummation of the g^2-power series, it is plausible to demand that the zeroth-order functions ($p = 0$) should possess *naive* asymptotic freedom, i.e.

$$\Gamma^{[r,0]}(\{\lambda k\}) \to \Gamma^{(0)\mathrm{pert}}(\{\lambda k\}), \qquad \lambda \to \infty\,, \tag{2.4}$$

as the set $\{k\}$ of external four-momenta are scaled up uniformly. These simple-looking conditions will be seen to strongly restrict the nonperturbative extension. We will actually impose a restriction slightly stronger than (2.4), namely the requirement that

$$\begin{gathered}\textit{the nonperturbative extension of } \Gamma^{(\mathrm{pert})} \\ \textit{should continue to be perturbatively renormalizable.}\end{gathered} \tag{2.5}$$

In an asymptotically free theory, and only there, this requirement is intuitively plausible: the large-momentum behavior of vertex functions in the loop integrals of (1.13) is known to be essentially the

perturbative one, apart from slowly varying logarithmic modifications that should not change the divergence pattern qualitatively. What makes (2.5) a somewhat stronger statement (for vertices with $N \geq 3$) is the implied condition that behavior no worse than for the perturbative vertex should obtain even when *only those momenta of* Γ *that run in a loop become large*, while the remaining, "external" ones are kept constant. Comparison between eqs. (3.8) and (3.9) below, for the three-gluon vertex, will pin down this difference more quantitatively in a specific example.

We already emphasized that in the "nonperturbative direction" of the sequence (2.1), characterized by the index r and relating to the dependence on Λ^2, the approximation, in contrast to the local one provided by a Taylor series around a point, must be *global*. On the other hand, at the basis of any perturbative renormalization process lies the possibility of *superficial convergence assessment by integer-power counting*. The only simple meeting point for these two requirements, and for the direction provided by the examples in sect. 1 of nonperturbative quantities, is the idea of representing $\Gamma^{[r,0]}$, the nonperturbatively modified vertex functions of zeroth perturbative order, by *rational approximants in* Λ^2 of increasing order r. This will indeed be seen to lead to a viable approximation scheme.

2.2 Gluonic Two-Point Function

In this section, we give details for the simplest case of a scalar vertex function with only one invariant momentum argument: the two-point vertex, or negative-inverse propagator, of transverse gluons. In the notation explained in sect. 1, this is

$$\Gamma_T(k^2) = -\frac{1}{D_T(k^2)}\,, \tag{2.6}$$

where D_T is the invariant function defined by the tensor decomposition of the full Euclidean gluon propagator,

$$D^{\mu\nu}(k) = t^{\mu\nu}(k) D_T(k^2) + l^{\mu\nu}(k) D_L\left(k^2\right)\,; \tag{2.7}$$

$$t^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = \delta^{\mu\nu} - l^{\mu\nu}(k)\,; \tag{2.8}$$

$$D_L(k^2) = \frac{\xi_0}{k^2}\,. \tag{2.9}$$

The propagation characteristics of the gluonic elementary excitation, $A_a^\mu(x)|0>$, will be determined by the zeroes and branch points of Γ_T in the complex k^2 plane. In the Euclidean domain, its rational approximants, which we will characterize by their *denominator* degrees r, will be of the form

$$\begin{aligned} -\Gamma_T^{[r,0]}(k^2,\Lambda^2) &= \frac{(k^2)^{r+1}+\zeta_{r,1}\Lambda^2(k^2)^r+\ldots+\zeta_{r,r+1}(\Lambda^2)^{r+1}}{(k^2)^r+\eta_{r,1}\Lambda^2(k^2)^{r-1}+\ldots+\eta_{r,r}(\Lambda^2)^r}\,; \\ r &= 0,1,2,3,\ldots\;; \end{aligned} \tag{2.10}$$

where the $2r+1$ coefficients $\zeta_{r,i}$ and $\eta_{r,j}$ are all real. Note how the boundary condition (2.4), in which

$$-\,\Gamma_T^{(0)\mathrm{pert}}(k^2) = k^2\,, \tag{2.11}$$

has uniquely fixed both the relative degrees and the leading coefficients of the numerator and denominator, and how condition (2.3) is then automatically fulfilled. *Only in an asymptotically free theory does one have such strong a priori restrictions on the form of the approximants.*

Without imposing specific dynamics, the sequence (2.10) still covers a variety of physical situations. We do not give a complete classification here, which would include several unphysical or exotic cases, but mention just the two subsequences of primary physical interest:

(1) "Particle" subsequence. Here r is even, so that the propagation function $D_T^{[r,0]}(k^2)$, by (2.6), has odd denominator degree $r+1$, and therefore at least one pole on the real k^2 axis, since the coefficients are real. (Note our convention of using the rational-approximation index r also on the propagator, although for the latter it gives the numerator, rather than the denominator, degree.) If the real pole closest to the origin sits at timelike Minkowskian (i.e. negative Euclidean) k^2, it represents a stable, asymptotically detectable gluon particle. Assuming – as everybody seems to have assumed tacitly since the classic papers of Lehmann and Källen [19] – that the elementary operator field can connect the vacuum to at most one single-particle state, one would expect this mass-shell pole position to stabilize as r is increased. The remaining r poles and r numerator zeroes of $D_T^{[r,0]}$ would then be expected to come again on the real axis, but separated from the particle pole, and for increasing r would be expected to settle into the alternating pattern that in the context of rational approximants is known [20] to approximate a branch cut – the Lehmann-Källen dressing cut, arising from virtual decays of the particle into multiparticle configurations (Fig. 1A).

The simplest approximant of this sequence, $r=0$, is

$$-\,\Gamma_T^{[0,0]}(k^2,\Lambda^2) = k^2 + \zeta_{0,1}\Lambda^2 \qquad (\zeta_{0,1} \text{ real and } > 0) \tag{2.12}$$

which when compared to (2.11) represents the general Schwinger mechanism [21]: the spontaneous creation of a mass $m^2 = \zeta_{0,1}\Lambda^2$ of the nonperturbative type (1.2) in a massless bare propagator.

(2) "Quasiparticle" subsequence. Here r is odd, so that $D_T^{[r,0]}$ has an even number $r+1$ of poles, and at least one real zero. Ignoring again the exotic possibility of two or more stable-gluon poles at different real masses, one would expect the two poles closest to the origin to come as a complex-conjugate pair at, say, $k^2 = -\sigma_{r,\pm}\Lambda^2$, with $\sigma_{r,+} = \sigma_{r,-}^*$ being a dimensionless complex number. As discussed in [22] and in sect. 5 below, the conceptual problems apparently caused by this complex-poles structure are not insurmountable, provided the solution is used consistently. The pair would represent an *intrinsically short-lived elementary excitation* of the gauge-vector field, with lifetime of the order of $1/\Lambda$. Note that the real zero closest to the origin – corresponding to a pole of the function

(2.6), i.e., a singular gluonic self-energy – is every bit as essential in this context as the complex pole pair, as it will be seen to make the connected Green's functions nonsingular in the invariant masses of external vector lines. The remaining even number $r-1$ both of poles and of zeroes may then come in complex-conjugate pairs, farther from the origin than the leading "quasiparticle" pair, with poles and zeroes interspersed so as to approximate two conjugate branch lines (Fig. 1B). These again would represent dressing – the short-lived excitation coupling dynamically to multiple copies of itself. The absence of real-axis branchpoints would signal that it has no channels for decay into stable fragments that would jointly carry the open quantum numbers of a gluon. This structure would fit most closely the empirical situation for the short-lived gluon presumed to be present at the origin of a gluon-jet event.

The simplest approximant of this sequence, $r=1$, is

$$\begin{aligned} -\Gamma_T^{[1,0]}(k^2,\Lambda^2) &= \frac{(k^2)^2+\zeta_{1,1}\Lambda^2 k^2+\zeta_{1,2}\Lambda^4}{k^2+\eta_{1,1}\Lambda^2} \\ &= k^2+u_{1,1}\Lambda^2+\frac{u_{1,3}\Lambda^4}{k^2+u_{1,2}\Lambda^2} \end{aligned} \tag{2.13}$$

with real coefficients $u_{1,i}$ given by

$$u_{1,1}=\zeta_{1,1}-\eta_{1,1}; \qquad u_{1,2}=\eta_{1,1}; \qquad u_{1,3}=\zeta_{1,2}-\eta_{1,1}u_{1,1};$$

and satisfying

$$u_{1,3} > \left[\frac{1}{2}(u_{1,1}-u_{1,2})\right]^2 . \tag{2.14}$$

The corresponding nonperturbatively modified propagator of zeroth perturbative order,

$$D_T^{[1,0]}(k^2,\Lambda^2) = \frac{k^2+u_{1,2}\Lambda^2}{(k^2+\sigma_{1,1}\Lambda^2)(k^2+\sigma_{1,3}\Lambda^2)}\,, \tag{2.15}$$

exhibits the minimum of features mentioned above: a real zero at $k^2=-u_{1,2}\Lambda^2$, and a "quasiparticle" pair of complex-conjugate poles at $k^2=-\sigma_{1,\pm}\Lambda^2$, where

$$\sigma_{1,+}=\sigma_{1,1}=\frac{1}{2}\,(u_{1,1}+u_{1,2})+i\sqrt{u_{1,3}-\left[\frac{1}{2}(u_{1,1}-u_{1,2})\right]^2}=\sigma_{1,3}^*=\sigma_{1,-}^* \tag{2.16}$$

At $r=1$, there are no further zeroes and poles as yet to simulate dressing cuts.

Eqs. (2.15) and (2.13) are of a form suggested in [22] (for $u_{1,2}=0$) and used in [7] as an element of an approximate, nonperturbative DS solution, and that form can now be identified as the lowest member of (the odd-r subsequence of) a systematic sequence (2.10) of approximants to the nonperturbatively modified function (2.6). A special case,

$$D_T^{(GZ)}(k^2) = \frac{k^2}{k^4+\gamma}\,, \tag{2.17}$$

with $u_{1,1} = u_{1,2} = 0$ and $u_{1,3}\Lambda^4 = \gamma$, had actually been arrived at much earlier by Gribov [14], and was later derived independently by Zwanziger [12], via the entirely different route of fundamental-domain restrictions. Because of this different origin, the Gribov-Zwanziger γ term is present *already at tree level*, i.e. in the analog of what we called $\Gamma^{(0)\mathrm{pert}}$, whereas the mechanism to be used in sect. 4 to stabilize nonzero $u_{1,i}$ coefficients in (2.13) will be seen to operate from DS *loops*.

It is clearly desirable to try out all of the above types of approximants in the DS equations, to determine those that can achieve dynamical self-consistency and, if necessary, to further distinguish between the latter by a stability analysis. We do not here embark on such a comprehensive study, but describe a few steps toward the much more restricted program of trying out the "quasiparticle" subsequence – with odd r and *only* complex-conjugate propagator singularities. In this subsequence, the nonperturbatively modified two-vector vertex in zeroth perturbative order will be of the form

$$\begin{aligned} -\Gamma_T^{[r,0]}(k^2,\Lambda^2) &= k^2 + u_{r,1}\Lambda^2 + \frac{u_{r,3}\Lambda^4}{k^2 + u_{r,2}\Lambda^2} \\ &\quad + \sum_{s=1}^{(r-1)/2} \left[\frac{u_{r,4s+1}\Lambda^4}{k^2 + u_{r,4s}\Lambda^2} + \frac{u_{r,4s+3}\Lambda^4}{k^2 + u_{r,4s+2}\Lambda^2} \right] \end{aligned} \tag{2.18}$$

$$r = 1, 3, 5 \ldots \tag{2.19}$$

generalizing eq. (2.13), with $u_{r,1}, u_{r,2}, u_{r,3}$ real, with

$$u_{r,4s+2} = u^*_{r,4s}, \qquad u_{r,4s+3} = u^*_{r,4s+1} \qquad \left(s = 1 \ldots \frac{r-1}{2}\right), \tag{2.20}$$

and such that all poles of the corresponding propagation function,

$$D_T^{[r,0]}(k^2) = \frac{k^2 + u_{r,2}\Lambda^2}{(k^2 + \sigma_{r,1}\Lambda^2)(k^2 + \sigma_{r,3}\Lambda^2)} \prod_{s=1}^{(r-1)/2} \frac{(k^2 + u_{r,4s}\Lambda^2)(k^2 + u_{r,4s+2}\Lambda^2)}{(k^2 + \sigma_{r,4s+1}\Lambda^2)(k^2 + \sigma_{r,4s+3}\Lambda^2)}, \tag{2.21}$$

form complex-conjugate pairs, with $\sigma_{r,1}, \sigma_{r,3}$ denoting the leading "quasiparticle" pair closest to the origin of the k^2 plane, i.e.,

$$\sigma_{r,3} = \sigma^*_{r,1}; \;\; \sigma_{r,4s+3} = \sigma^*_{r,4s+1}; \quad |\sigma_{r,4s+1}| > |\sigma_{r,1}| \quad (s = 1 \ldots \frac{r-1}{2}). \tag{2.22}$$

The naive-asymptotic-freedom condition built into (2.10), $\Gamma_T^{[r,0]} \to -k^2$ for $k^2 >> \Lambda^2$, guarantees that in the pole decomposition,

$$D_T^{[r,0]}(k^2) = \frac{\rho_{r,0}}{k^2 + \sigma_{r,1}\Lambda^2} + \frac{\rho_{r,2}}{k^2 + \sigma_{r,3}\Lambda^2} + \sum_{s=1}^{(r-1)/2} \left[\frac{\rho_{r,4s}}{k^2 + \sigma_{r,4s+1}\Lambda^2} + \frac{\rho_{r,4s+2}}{k^2 + \sigma_{r,4s+3}\Lambda^2} \right], \tag{2.23}$$

the sum of the dimensionless residues is unity:

$$\rho_{r,0} + \rho_{r,2} + \sum_{s=1}^{(r-1)/2} (\rho_{r,4s} + \rho_{r,4s+2}) = 1 \quad (r = 1, 3, 5 \ldots). \tag{2.24}$$

This, incidentally, is one of the features that distinguish the functions (2.23) from propagators plagued by so-called ghost poles, which – usually as a result of inadequate approximations – vexed field theorists in the 1950's [23], and for which the sum-of-residues was negative or vanishing.

2.3 Relation with the OPE

To exhibit the connections of the sequential approximation (2.10) with an established area of QCD methodology, we briefly look at the operator-product expansion (OPE). The OPE for the *elementary* QCD fields has only recently begun to be established correctly (for recent results and literature, see refs. [24] through [27]) and is not commonly discussed in the present terms, yet it already represents a step towards describing the "truly nonperturbative" Λ^2 dependence. Again we consider the simplest case of the one-variable vertex function (2.6), and restrict ourselves to a theory with at most massless quarks, so as not to have to worry about the role of other invariant mass scales besides Λ. By writing an OPE for the Euclidean two-point function of the gauge field, transforming to momentum space, contracting with the $t^{\mu\nu}$ of eq. (2.8) to project out the transverse portion, and forming the inverse function (2.6), one arrives at an expansion of the general form

$$\begin{aligned} -\Gamma_T(k^2; g^2(\nu); \nu) \quad = \quad & k^2 \left\{ 1 + \sum_{p=1}^{\infty} \left[\tfrac{g(\nu)}{4\pi}\right]^{2p} L_{0,p}\left(\tfrac{k^2}{\nu^2}\right) \right\} \\ & + \Lambda^2 \left\{ L_{10} + \sum_{p=1}^{\infty} \left[\tfrac{g(\nu)}{4\pi}\right]^{2p} L_{1,p}\left(\tfrac{k^2}{\nu^2}\right) \right\} \\ & + \tfrac{\Lambda^4}{k^2} \left\{ L_{20} + \sum_{p=1}^{\infty} \left[\tfrac{g(\nu)}{4\pi}\right]^{2p} L_{2,p}\left(\tfrac{k^2}{\nu^2}\right) \right\} \\ & + \dots \\ & + \tfrac{(\Lambda^2)^n}{(k^2)^{n-1}} \left\{ L_{n,0} + \dots \right\} \\ & + \dots \end{aligned} \tag{2.25}$$

where the coefficient functions (modified Wilson coefficients) are $p-th$ degree polynomials of logarithms:

$$L_{n,p} = c_{n,p}^{(0)} + c_{n,p}^{(1)} \ln\left(\frac{k^2}{\nu^2}\right) + \dots + c_{n,p}^{(p)} \left[\ln\left(\frac{k^2}{\nu^2}\right)\right]^p . \tag{2.26}$$

This may be viewed as a form of the series (2.1/2.2) where the $\Gamma^{[r,p)}$ have in turn been expanded in power series in Λ^2. More precisely, apart from the typical perturbative logarithms, it is an expansion (presumably semi-convergent) in powers of (Λ^2/k^2) for $k^2 >> \Lambda^2$, the deep-Euclidean limit. The

first line of (2.25) is identical with the perturbation series (1.17), so the boundary condition (2.3) is satisfied. The presence of the other terms, containing powers of the g^2-nonanalytic scale (1.2), shows clearly that the perturbative series alone would be an incomplete solution even when summed to all orders. At the core of these additional terms are the quantities $L_{n0}(\Lambda^2)^n, n \geq 1$, which the OPE derivation identifies as linear combinations of vacuum expectation values of composite operators ("vacuum condensates") of increasing mass dimension, $2n$.

As it stands, the OPE (2.25) does not satisfy our needs, for two reasons. First, the OPE by itself does not determine the quantities L_{n0} – this requires a truly dynamical principle, such as the DS equations. Second, even if the L_{n0} were determined dynamically up to some n, no finite order of (2.25) would be a satisfactory continuation of the vertex function into the region of primary physical interest – the region $k^2 \leq \Lambda^2$ of typical hadronic masses. In this region, given our assumption that $[g(\nu)/4\pi]^2$ never becomes large, the important task clearly is to obtain a continuation-through-resummation of the "vertical" sums in (2.25) – in particular, the $p = 0$ vertical sum,

$$-\Gamma_T^{[r,0]}(k^2, \Lambda^2) = k^2 \left[1 + L_{10}\frac{\Lambda^2}{k^2} + L_{20}\left(\frac{\Lambda^2}{k^2}\right)^2 + \ldots\right] , \tag{2.27}$$

which is *free of perturbative logarithms.* (Once this is done, the remaining ($p \geq 1$) "vertical" summations can in principle be generated iteratively from $\Gamma^{[r,0]}$ through the DS equations, just as in the perturbative case.) From this standpoint, the rational approximations (2.10) may now be viewed as a systematic sequence of continuations-through-resummation of the OPE subseries (2.27), which locate its low-k^2 zeroes and singularities with increasing accuracy.

We stress that the OPE (2.25) has been used here only as a point of comparison. The "extended-iterative" scheme differs from it in more than the technical aspect of parametrization. (In the OPE, the fundamental parameters are an infinite set of dimensionful vacuum condensates, whereas in (2.10) that role is played by the dimensionless vertex coefficients such as $(\zeta_{r,i}, \eta_{r,j})$, and condensates are secondary quantities calculable in principle in terms of the latter – see the second of refs. [7] for $[r, p] = [1, 0]$ examples). Its properties can be qualitatively different from any finite order of (2.25) because the two are separated by a nontrivial step of analytic continuation. In particular, the "horizontal" sums in (2.25) are always ordinary QCD perturbation series based on the Feynman rules, with all the attendant problems arising from the empirically wrong zeroth-order spectrum (including free, massless, physical gluons, and their associated infrared singularities) of the latter. By contrast, expansion (2.1/2.2) is based on the use of (inter alia) modified propagators such as (2.21), which in particular produce no infrared singularities at all (the unphysical longitudinal gluons still do, but their effects are always preceded by ξ factors that identify them as gauge-fixing artefacts). This opens up the possibility that Borel transforms with respect to g^2 of eq. (2.2) may have, apart from gauge-fixing artefacts, no genuine infrared renormalons, the effects of the latter having been absorbed in a redefinition of the

zeroth perturbative order.

3 Three-Vector-Vertex Approximants

The hierarchical structure of the DS equations (1.13) implies that in principle all proper vertices Γ_N should be treated simultaneously by mutually consistent, nonperturbative approximants. The method described here will however be found to have the simplifying feature that the essential self-consistency problem – for the vertices $\Gamma^{[r,0]}$ – is *rigorously* restricted to the small finite set (1.14) of superficially divergent vertices. In this section we therefore compile formulas for rational functions $\Gamma^{[r,0]}_{sdiv}$, concentrating on the example of the proper three-vector vertex Γ_{3V}, which serves to illustrate all the essential features. The largely analogous material for the remaining Γ_{sdiv} vertices will be relegated to appendix A. The only new aspect this discussion will turn up will be the "factorizing-denominator" rule discussed in connection with eq. (3.6) below. Otherwise, the main complications will be the multi-variable nature, and the notoriously unwieldy color-and-Lorentz-tensor structure, of the $N \geq 3$ QCD vertices. One measure we will adopt to control this purely kinematical complexity is to restrict ourselves to vertices with at most transverse external gluon lines, wich are sufficient for performing calculations in the Landau ($\xi_0 = 0$) gauge. As shown most clearly by the example of eq. (2.9), it is in these that nonperturbative effects are expected to develop most freely. Amplitudes with at least one longitudinal gluon are strongly restricted by the Slavnov-Taylor (ST) identities, whose nonperturbative saturation is not a subject of this paper.

The three-gluon vertex has color structure,

$$(\Gamma_{3V})^{\mu\lambda\nu}_{abc} = f_{abc}\Gamma^{\mu\lambda\nu}_{3V(f)} + d_{abc}\Gamma^{\mu\lambda\nu}_{3V(d)}\,, \tag{3.1}$$

in terms of f-type (antisymmetric) and d-type (symmetric) structure constants. Each color component in turn is of the general Lorentz structure discussed by Ball and Chiu [28]: a linear combination of 14 independent third-rank tensors, with 6 different invariant functions having appropriate symmetry or antisymmetry properties in the Lorentz-scalar variables p_1^2, p_2^2, p_3^2. We consider only the totally transverse portions,

$$\Gamma^{\mu'\lambda'\nu'}_{3T(c)} = t^{\mu'\mu}(p_1)t^{\lambda'\lambda}(p_2)t^{\nu'\nu}(p_3)\Gamma^{\mu\lambda\nu}_{3V(c)} \qquad (c = f \text{ or } d)\,, \tag{3.2}$$

which have contributions from only 4 Lorentz tensors combined with two different invariant functions F_0, F_1 for each color component:

$$\Gamma^{\mu'\lambda'\nu'}_{3T(c)}(p_1,p_2,p_3) = t^{\mu'\mu}(p_1)t^{\lambda'\lambda}(p_2)t^{\nu'\nu}(p_3)\times$$

$$\times\left\{\delta^{\lambda\nu}(p_2-p_3)^{\mu}F_{(c)0}(p_2^2,p_3^2,p_1^2)\right.$$

$$+\,\delta^{\mu\nu}(p_3-p_1)^\lambda F_{(c)0}(p_3^2,p_1^2,p_2^2) \tag{3.3}$$
$$+\delta^{\mu\lambda}(p_1-p_2)^\nu F_{(c)0}(p_1^2,p_2^2,p_3^2)$$
$$+(p_2-p_3)^\mu(p_3-p_1)^\lambda(p_1-p_2)^\nu F_{(c)1}(p_1^2,p_2^2,p_3^2)\Big\}\,; \qquad (c=f \text{ or } d)\,.$$

Remember $p_1+p_2+p_3=0$. Here the dimensionless functions $F_{(c)0}$ are symmetric (for $c=f$) or antisymmetric (for $c=d$) in their first two arguments, while the functions $F_{(c)1}$, of mass dimension -2, are totally symmetric ($c=f$) or antisymmetric ($c=d$) in all three arguments. The perturbative zeroth-order limits are,

$$F_{(c)k}^{(o)\mathrm{pert}}=\delta_{cf}\delta_{k0} \qquad (c=f \text{ or } d, \quad k=0 \text{ or } 1)\,. \tag{3.4}$$

In setting up sequences of rational approximants for these four invariant functions, capable of dynamical consistency with the gluon-propagator sequence (2.18), one encounters a new aspect: approximants with the most general denominator polynomials in the three variables p_1^2,p_2^2,p_3^2 are not useful. The zeroes of such a general denominator in any one variable p_i^2 are complicated non-rational functions of the two other variables, and this will stand in the way of DS self-consistency when a vertex transfers its structure in an external momentum to the next lower vertex through the hierarchical coupling. At the expense of slower convergence of the approximating sequences, we must restrict ourselves to the narrower but still sufficiently general class of *factorizing-denominator rational approximants (FDRA)*, i.e. those in which the denominator factorizes with constant zeroes in all three variables. To see that such a more special approximation is always possible in principle, consider e.g. the function $F_{(f)0}(p_1^2,p_2^2;p_3^2)$ as the kernel of a symmetric integral operator parametrically dependent on p_3^2, and write an eigenfunction expansion

$$F_{(f)0}(p_1^2,p_2^2;p_3^2)=\sum_n g_n(p_3^2)f_n(p_1^2)f_n(p_2^2)\,, \tag{3.5}$$

with eigenvalues g_n. By using single-variable rational approximation for the f_n and g_n functions, letting n range over finite but increasing numbers of eigenvalues, suitably discretizing the integral in case there is a continuous spectrum, and putting everything over a common denominator, one generates a sequence of approximants in which denominators have the desired, fully factorized form. Similar considerations apply to the other invariant functions.

By extension of our above definition, the degree r of rational approximation for the three-point vertex $\Gamma_{3T}^{[r,0]}$ in zeroth perturbative order will be the number of different denominator zeroes in any one variable *for the entire tensorial vertex.* That is, to order $p=0$ in eq. (2.1) and degree r, all invariant functions will be of the form

$$F_{(c)k}^{[r,0]}(p_1^2,p_2^2,p_3^2)=\frac{N_{3V(c)k}^{(r)}(p_1^2,p_2^2,p_3^2)}{\left[\prod\limits_{s=1}^{r}\left(p_1^2+u'_{r,2s}\Lambda^2\right)\right]\left[\prod\limits_{s=1}^{r}\left(p_2^2+u'_{r,2s}\Lambda^2\right)\right]\left[\prod\limits_{s=1}^{r}\left(p_3^2+u'_{r,2s}\Lambda^2\right)\right]} \tag{3.6}$$

$$(c = f \text{ or } d; \quad k = 0 \text{ or } 1; \quad r = 1, 3, 5, \ldots)$$

with the same fully factorized denominator, where the appropriate mass dimensions and Bose – symmetry restrictions, as well as the boundary conditions (2.3/2.4), are built into the numerator polynomials

$$N^{(r)}_{3V(c)k}\left(p_1^2, p_2^2, p_3^2\right) = \delta_{cf}\delta_{k0}\left(p_1^2 p_2^2 p_3^2\right)^r + \sum_{m_1, m_2, m_3 \geq 0} C^{(c)k}_{m_1 m_2 m_3}\left(p_1^2\right)^{m_1}\left(p_2^2\right)^{m_2}\left(p_3^2\right)^{m_3}\left(\Lambda^2\right)^{3r-k-(m_1+m_2+m_3)} . \tag{3.7}$$

Condition (2.4) allows nonzero coefficients $C_{m_1 m_2 m_3}$ only for

$$m_1 + m_2 + m_3 \leq 3r - k - 1 . \tag{3.8}$$

Additional restrictions follow from the postulate (2.5) of preservation of perturbative renormalizability. As a minimum, it requires that no part of Γ_{3V} should lead to ultraviolet divergences stronger than the corresponding perturbative ones. Since in a $1PI$ diagram one of the three legs of Γ_{3V} may be external, we conclude that when any two of its momenta are running in a loop, the vertex should behave no worse than q^1 at large loop momenta q. For the invariant functions (3.6) this requires

$$m_1 + m_2 \leq 2r - k, \quad m_2 + m_3 \leq 2r - k, \quad m_3 + m_1 \leq 2r - k . \tag{3.9}$$

Moreover, the coefficients should exhibit the symmetries with respect to m_1, m_2, m_3 necessary to fulfill the conditions of partial or total (anti-)symmetry listed before.

Comparison between eqs. (3.9) and (3.8) illustrates well the statements made after eq. (2.5): of the conditions required for a general rational function to reduce to one that preserves perturbative power counting, the vast majority are already enforced by the asymptotic-freedom condition (3.8), with (3.9) representing only a mild additional restriction. (In fact, on the $r = 1$ level written out below, (3.9) will already be fully implied by (3.8).) For the discussion of sect. 4, the essential and simple consequence of the factorizing-denominator structure is that with respect to any one argument p_k^2, the full $p = 0$ vertex has a partial-fraction decomposition

$$\begin{aligned}\left[\Gamma^{\mu_1\mu_2\mu_3}_{3T(c)}(p_1, p_2, p_3)\right]^{[r,0]} &= B^{\mu_i\mu_j\mu_k}_{(c),0,r}(p_i, p_j, p_k) \\ &+ \sum_{s=1}^{r} B^{\mu_i\mu_j\mu_k}_{(c)s,r}(p_i, p_j)\left(\frac{\Lambda^2}{p_k^2 + u'_{r,2s}\Lambda^2}\right) \\ &(i, j, k = 1, 2, 3 \text{ and cyclic}) ,\end{aligned} \tag{3.10}$$

where invariant functions for the residue tensors $B_{s,r}$ with $s \geq 1$ depend only on p_i^2 and p_j^2, whereas the regular part $B_{0,r}$ is in addition an r-th order polynomial in p_k^2. (Note that with respect to any *single* p_k^2, the conditions (3.8/3.9) do not rule out terms that grow polynomially. Without dwelling on this point, we remark that most of these terms will however turn out to vanish on dynamical grounds.)

The question may be raised as to whether the choice of variables p_1^2, p_2^2, p_3^2 in constructing the FDRA of eq. (3.6) is unique. Could other sets of Lorentz invariants be used, such as $\{p_1 \cdot p_2, p_2 \cdot p_3, p_3 \cdot p_1\}$? In eq. (3.6), we have, for brevity, anticipated the fact that dynamical consistency between the Γ_{2V} and Γ_{3V} vertices will indeed require that FDRA's for Γ_{3V} be constructed in the p_k^2 variables.

To illustrate these structures, we write expressions (3.6), now fully decomposed into partial fractions in analogy with eq. (2.18), for the simplest case, $r = 1$. In this case, all rational structure can be expressed in terms of the single-variable pole factors

$$\Pi_i = \frac{\Lambda^2}{p_i^2 + u'_{1,2}\Lambda^2} \qquad (i = 1, 2, 3)\,. \tag{3.11}$$

The f-type invariant-function approximants then read,

$$\begin{aligned} F^{[1,0]}_{(f)0}\left(p_1^2, p_2^2; p_3^2\right) = \;& 1 + x_{1,1}\left(\Pi_1 + \Pi_2\right) + x_{1,3}\Pi_3 + \left(x_{1,2} + x'_{1,2}\tfrac{1}{\Pi_3}\right)\Pi_1\Pi_2 \\ & + \left[\left(x_{1,4} + x'_{1,4}\tfrac{1}{\Pi_2}\right)\Pi_1 + \left(x_{1,4} + x'_{1,4}\tfrac{1}{\Pi_1}\right)\Pi_2\right]\Pi_3 \\ & + x_{1,5}\left(\Pi_1\Pi_2\Pi_3\right)\,, \end{aligned} \tag{3.12}$$

$$F^{[1,0]}_{(f)1}\left(p_1^2, p_2^2, p_3^2\right) = \frac{1}{\Lambda^2}\left[x_{1,6}\left(\Pi_1\Pi_2 + \Pi_2\Pi_3 + \Pi_3\Pi_1\right) + x_{1,7}\left(\Pi_1\Pi_2\Pi_3\right)\right]\,, \tag{3.13}$$

whereas the d-type approximants are,

$$F^{[1,0]}_{(d)0}\left(p_1^2, p_2^2; p_3^2\right) = x_{1,8}\left(\Pi_1 - \Pi_2\right) + \left[\left(x_{1,9} + x'_{1,9}\frac{1}{\Pi_2}\right)\Pi_1 - \left(x_{1,9} + x'_{1,9}\frac{1}{\Pi_1}\right)\Pi_2\right]\Pi_3\,, \tag{3.14}$$

$$F^{[1,0]}_{(d)1}\left(p_1^2, p_2^2, p_3^2\right) \equiv 0\,. \tag{3.15}$$

The latter result reflects the fact that for $k = 1$ and $r = 1$, no fully antisymmetric numerator polynomial in three variables obeying restriction (3.8) can be constructed.

To order $(r = 1, p = 0)$, the set of twelve dimensionless coefficients,

$$x_1 = \{x_{1,1}, x_{1,2}, \ldots x'_{1,9}\}\,, \tag{3.16}$$

which must be real numbers of order unity, are the fundamental vertex parameters to be determined from the DS equations (1.13). Note that all invariant functions at this level feature one single pole position, $-u'_{1,2}\Lambda^2$. While e.g. in the $k = 0$ functions the partial (anti-) symmetry would seem to permit the use of two different pole positions, such an approximant, under our above definition, would already be of degree $r = 2$. This strict definition of r may at first appear to be overly rigid, but it will turn out to be the relevant one for the self-consistency problem.

4 Dyson-Schwinger self-consistency

When introduced into the DS equations, the nonperturbative sequence (2.1), in contrast to the purely perturbative one, encounters a nontrivial self-consistency problem – the self-reproduction of the $p = 0$ functions $\Gamma^{[r,0]}$ at a given level r. For the first iteration, and in the notation of eq. (1.13), this problem can be stated as

$$\left\{ \left(\frac{g_0}{4\pi} \right)^2 \Phi_N \left[\Gamma^{[r,0]} \right] \right\}_{R,\nu} = \left[\Gamma_N^{[r,0)} - \Gamma_N^{(0)\mathrm{pert}} \right] + 0 \left(g^2(\nu), e(r+1) \right) . \tag{4.1}$$

Here, $e(r+1)$ stands for the approximation error in the r direction which, in contrast to that of the semiconvergent p sequence, cannot be characterized by a power of some expansion parameter.

It is useful to reflect at this point on what the precise meaning can be of "solving a Dyson-Schwinger system by a sequence $\Gamma^{[r,p]}$":

(1) Upon iteration of the system around the starting point $\Gamma^{[r,0]}$, the sequence should reproduce itself at each iteration step *up to corrections of the next higher perturbative order* p, which are simultaneously generated in the process.

(2) Self-reproduction of $\Gamma_N^{[r,0]}$, which features a number $n(r)$ of dimensionless nonperturbative parameters, can only mean that this approximant is made to coincide with the r.h.s. of its own DS equation at $n(r)$ points in the space of scalar momentum variables k^2 of Γ_N, or more generally (since global approximants allow a variety of matching prescriptions) *with respect to* $n(r)$ *"comparison data"* (function values, derivatives, pole positions, integrated quantities ...). At all other points, or in all other data, a matching error will necessarily remain that can be improved only by going to the next higher r.

4.1 Transverse-Gluon Self-Energy

It is characteristic of the scheme discussed here that the hardest problem for direct numerical Dyson-Schwinger solutions – the self-reproduction of *momentum* structure – has a relatively straightforward and explicit answer: the $(N' > N)$-point vertices entering the dressing functional Φ will "hand down" their rational structure in external momenta to the N-point ones. This pattern may at first sight appear simplistic, but in fact turns out to be an efficient way of exploiting the most difficult and peculiar structural feature of a Dyson-Schwinger system – the hierarchical coupling. It has, however, the immediate consequence that the set of nonperturbatively modified vertices can only be treated *as a whole*: no self-consistency, however approximate, will be possible in this framework if one seeks solutions to particular DS equations while treating the higher vertices appearing there by more or less unrelated assumptions.

Less straightforward is self-reproduction of the *coupling* structure. This problem is already clearly visible in (4.1): the DS loop integrals constituting Φ_N must produce the bracket on the r.h.s., with no g^2 prefactor, in spite of the fact that they always come with at least one g_0^2 prefactor. The importance of securing this feature can hardly be overstated: since perturbative corrections of finite order ($p \geq 1$) cannot alter the qualitative spectral properties of a solution, the essential nonperturbative features, and in particular qualitative changes expected in the spectrum of elementary excitations – i.e., in the two-point functions of the basic fields – must establish themselves already on the $p = 0$ level.

The mathematical mechanism leading to the "eating of g_0^2 prefactors", a low-order version of which was used already in [22] and discussed in detail in [7], will again be demonstrated for the example of the one-variable function (2.6). Its DS equation,

$$\Gamma_T(k^2) = -k^2 + \left(\frac{g_0}{4\pi}\right)^2 \sum_{M=A...F} \Phi_{TT}^{(M)} \left[\Gamma_2, \Gamma_3, \Gamma_{VVVT}\right] , \tag{4.2}$$

is stated diagrammatically in fig. 2, showing the one-DS-loop ($M = A, B, C, D$) and two-DS-loop terms ($M = E, F$) of the Φ_{TT} functional. (The term "DS loop" will be used to refer to the specific structure of DS integrals which are neither bare nor fully dressed, containing always one bare vertex times a number of dressed functions). Start at the one-loop level, where the terms $M = E, F$ do not yet contribute. For eq. (4.1), we have to evaluate

$$\begin{aligned}\Phi_{TT}^{(l=1)}\left[\Gamma^{[r,0]}\right] = \ & \Phi_{TT}^{(A)}\left[D^{[r,0]}, \Gamma_{VTV}^{[r,0]}\right] + \Phi_{TT}^{(B)}\left[\tilde{D}^{[r,0]}, \Gamma_{GT\overline{G}}^{[r,0]}\right] \\ & + \Phi_{TT}^{(C)}\left[D^{[r,0]}\right] + \sum_F \Phi_{TT}^{(D)}\left[S_F^{[r,0]}, \Gamma_{FT\overline{F}}^{[r,0]}\right] .\end{aligned} \tag{4.3}$$

Now use the FDRA structure of the $[r, 0]$ three-point vertices, as expressed in the partial-fraction decompositions (3.10), (A.24), and (A.44). Expression (4.3) then decomposes into terms regular and singular with respect to the external variable k^2:

$$\Phi_{TT}^{(l=1)}\left[\Gamma^{[r,0]}\right] = I_0^{(r)}(k^2) + \sum_{s=1}^{r} I_s^{(r)}(k^2) \left(\frac{\Lambda^2}{k^2 + u'_{r,2s}\Lambda^2}\right) . \tag{4.4}$$

The quadratically divergent loop integrals $I_s^{(r)}$ are of the same form as (4.3), but with the replacements

$$\Gamma_{VTV}^{[r,0]} \longrightarrow B_{(f)s}^{(r)} \quad \text{in} \quad \Phi_{TT}^{(A)} , \tag{4.5}$$

$$\Gamma_{GT\overline{G}}^{[r,0]} \longrightarrow \tilde{B}_{(f)s}^{(r)} \quad \text{in} \quad \Phi_{TT}^{(B)} , \tag{4.6}$$

$$\Gamma_{FT\overline{F}}^{[r,0]} \longrightarrow C_{(F)s}^{(r)} \quad \text{in} \quad \Phi_{TT}^{(D)} . \tag{4.7}$$

Indices (f) on the r.h. sides of (4.5/4.6) recall the fact that only the f_{abc} parts of those vertices contribute here. By definition, the "tadpole" contribution $\Phi_{TT}^{(C)}$, (which does not vanish as in the perturbative case), is entirely included in $I_0^{(r)}$, as it has no k^2-singular terms, and in fact is a constant.

In eq. (4.4), since we have treated the 3-point vertices consistently, i.e. by FDRA's of the same level r, the r.h.s. already displays the needed number of poles, and inspection of the $I_s^{(r)}$ integrals shows that they produce logarithmic branch points (generally complex since we are using the complex-pole propagators of the odd-r sequence) but no poles in k^2 of their own. It is then natural to choose the r positions and r residues of those poles as comparison data – if the poles were not matched, the matching error would be locally infinite. That is, one requires

$$u_{r,2s} = u'_{r,2s} \quad (s = 1 \ldots r)\,; \tag{4.8}$$

$$-\; u_{r,2s+1}\Lambda^2 = \left(\frac{g_0}{4\pi}\right)^2 I_s^{(r)}\left(-u_{r,2s}\Lambda^2\right) \quad (s = 1 \ldots r)\,. \tag{4.9}$$

Eq. (4.8) expresses the "handing down" of momentum structure, rationally approximated, from the 3-point vertices to the 2-point one under consideration. It immediately implies, by eq. (2.21), that the propagator will have *zeroes* at the positions of the gluon-variable poles of the 3-point vertices.

It would seem that as a $(2r+1)$-th comparison datum to fix the one still undetermined coefficient $u_{r,1}$ of (2.18), we could use the value of the smooth-remainder function,

$$J_0^{(r)}(k^2) = I_0^{(r)}(k^2) + \Lambda^2 \sum_{s=1}^{r} \frac{I_s^{(r)}(k^2) - I_s^{(r)}(-u_{r,2s}\Lambda^2)}{k^2 + u_{r,2s}\Lambda^2}\,, \tag{4.10}$$

at some arbitrary point, $k^2 = -\overline{u}\Lambda^2$, of the k^2 plane:

$$-\; u_{r,1}\Lambda^2 = \left(\frac{g_0}{4\pi}\right)^2 J_0^{(r)}(-\overline{u}\Lambda^2)\,. \tag{4.11}$$

We will soon see that the arbitrariness introduced at this point is only apparent. In eqs. (4.9) and (4.11), the problem of how to "eat" the g_0^2 prefactor to produce a $p = 0$ quantity is still with us.

4.2 The self-consistency mechanism

Evaluating now the I_s and J_0 integrals in dimensional regularization – note that all these integrals, because of the rational structure of integrands, can be evaluated by standard methods –, one finds

$$I_s^{(r)}(-u_{r,2s}\Lambda^2) = \left(\frac{\Lambda_\epsilon^2}{\nu_0^2}\right)^{-\epsilon} \left\{a_s(u,v,w,x,y,z)\cdot\frac{1}{\epsilon} + [\text{terms finite as } \epsilon \to 0] + 0(\epsilon)\right\}\Lambda^2 \quad (s = 1 \ldots r)\,, \tag{4.12}$$

$$J_0^{(r)}(-\overline{u}\Lambda^2) = \left(\frac{\Lambda_\epsilon^2}{\nu_0^2}\right)^{-\epsilon} \left\{a_0(u,v,w,x,y,z)\cdot\frac{1}{\epsilon} + [\text{terms finite as } \epsilon \to 0] + 0(\epsilon)\right\}\Lambda^2\,, \tag{4.13}$$

where the notation recalls that the Λ scale, too, should be continued to $D = 4 - 2\epsilon$ through the replacement $\beta(g') \to \beta(g') - \epsilon g'$ in its definition (1.2). The sets $u \ldots z$ are the dimensionless-coefficient sets of the $[r, 0]$ propagators and 3-point vertices, as defined in sect. 3 and appendix A, and include, for the time being, the quantity $\overline{u}$ of eq. (4.11).

To analyze the product $(g_0/4\pi)^2(\Lambda_\epsilon^2/\nu_0^2)^{-\epsilon}(1/\epsilon)$ now appearing in the self-consistency equations, we next impose the condition, inherent in (2.5), that perturbative coupling-constant renormalization, eq. (1.19), should apply. Then this product becomes,

$$\left[\frac{g(\nu)}{4\pi}\right]^2 Z_\alpha\left(g^2(\nu),\frac{1}{\epsilon}\right)\left(\frac{\Lambda_\epsilon^2}{\nu^2}\right)^{-\epsilon}\frac{1}{\epsilon}\equiv\Pi(\epsilon,g^2)\,. \tag{4.14}$$

To get a feeling for its structure, we first use a slightly sloppy argument [7,6] which, however, happens to convey the essential points. Start from

$$\left(\frac{\Lambda_\epsilon^2}{\nu^2}\right)^{-\epsilon}=1-\epsilon\ln\left(\frac{\Lambda^2}{\nu^2}\right)+O(\epsilon^2)\,, \tag{4.15}$$

which gives

$$\frac{1}{\epsilon}\left(\frac{\Lambda_\epsilon^2}{\nu^2}\right)^{-\epsilon}=-\ln\left(\frac{\Lambda^2}{\nu^2}\right)+\frac{1}{\epsilon}+O(\epsilon)\,. \tag{4.16}$$

The logarithmic term survives the removal-of-regulator limit, $\epsilon\to0$, in the same, familiar way as do the perturbative logarithms of, say, eq. (2.25), but it differs from them in one important respect – it leads them by one order in the perturbative g^{2p} classification. From eq. (1.3), at the one-loop level

$$-\ln\left(\frac{\Lambda^2}{\nu^2}\right)=\left[\frac{4\pi}{g(\nu)}\right]^2\frac{1}{\beta_0}\left\{1+O(g^2\ln g^2,g^4)\right\}\,. \tag{4.17}$$

This is precisely the $1/g^2$ factor needed to "eat" the overall g^2 factor in (4.14); it has survived due to its association with the $1/\epsilon$ divergence. We now have

$$\left[\frac{g(\nu)}{4\pi}\right]^2\frac{1}{\epsilon}\left(\frac{\Lambda^2\epsilon}{\nu^2}\right)^{-\epsilon}=\frac{1}{\beta_0}\left\{1+\left[\frac{g(\nu)}{4\pi}\right]^2\beta_0\frac{1}{\epsilon}+O(g^2\ln g^2,g^4)+O(g^2\epsilon)\right\}\,. \tag{4.18}$$

But Z_α, to the same order, is given by (1.20), so that to the order calculated, the divergences actually cancel in the product (4.14):

$$Z_\alpha\left\{\left[\frac{g(\nu)}{4\pi}\right]^2\frac{1}{\epsilon}\left(\frac{\Lambda_\epsilon^2}{\nu^2}\right)^{-\epsilon}\right\}=\frac{1}{\beta_0}\left[1+O(g^2\ln g^2)+O(g^4)+O(\epsilon)\right] \tag{4.19}$$

Although the terms denoted $O(g^4)$ do include (as do those in (1.20)) terms of the form g^4/ϵ, they are not to be kept in a one-loop calculation. We therefore find that not only do the nonperturbative parts of (2.18) – the r pole terms and the mass-type constant term – establish themselves in order $p=0$ in spite of the $p=1$ prefactor, but they do so without divergences to the order calculated, so that in particular *no nonlocal counterterms are needed* for the pole terms. This crucial prerequisite of perturbative renormalization is therefore automatically preserved.

The sloppiness of the above derivation based on 1-loop results lies hidden in the terms summarily denoted "$O(\epsilon^2)$" in (4.15) and "$O(g^4)$" in (1.20). When examining these more closely, one realizes

that $(\Lambda_\epsilon^2/\nu^2)^{-\epsilon}$ actually has an infinite subseries of terms of type $(\epsilon/g^2)^m, m \geq 1$, whereas Z_α has a subseries of terms of type $(g^2/\epsilon)^n, n \geq 0$, and the desired terms of type $\epsilon^0(g^2)^0$ in (4.14) can therefore be produced in an infinite number of ways. To account for all of these, one could obtain both subseries exactly by resummations of the LL type, but the result, while perfectly sufficient for the one-loop calculation, would not display the basic simplicity of the situation: the quantity Π of (4.14) is actually independent of g^2. This follows immediately from the exact integral representations of the two main factors. From the extension of definition (1.2) to $\epsilon \neq 0$, one finds by simple manipulations that

$$\left(\frac{\Lambda_\epsilon^2}{\nu^2}\right)^{-\epsilon} = \frac{\kappa_1}{\kappa(\nu)} \exp\left\{\int\limits_{\kappa_1}^{\kappa(\nu)} \frac{d\lambda}{\lambda + \epsilon f(\lambda)}\right\}, \tag{4.20}$$

where $\kappa = [g(\nu)/4\pi]^2$, and

$$f(\kappa) = \left[\beta_0 + \beta_1\kappa + \beta_2\kappa^2 + \ldots\right]^{-1} = \frac{-g\kappa}{\beta(g)}. \tag{4.21}$$

The $1/\kappa$ factor of (4.20) is the exact counterpart of the one of (4.17) and is what is needed for the "eating" process. The lower integration limit κ_1, fixed conventionally as part of the definition of the renormalization scheme, will in general depend on ϵ,

$$\kappa_1 = \kappa_1(\epsilon) = \kappa_1(0)[1 + O(\epsilon)], \tag{4.22}$$

with $\kappa_1(0)$ having the meaning of a renormalized coupling at the scale Λ. On the other hand, differentiating (1.19) with respect to ν leads to the relation

$$\kappa\frac{d}{d\kappa} \ln Z_\alpha(\kappa(\nu), \epsilon) = -\frac{1}{1 + \frac{\epsilon}{\kappa} f(\kappa)}, \tag{4.23}$$

whose reintegration *with respect to* κ under the initial condition $Z_\alpha(0, \epsilon) = 1$ gives

$$Z_\alpha(\kappa(\nu), \epsilon) = \exp\left\{-\int\limits_0^{\kappa(\nu)} \frac{d\lambda}{\lambda + \epsilon f(\lambda)}\right\}, \tag{4.24}$$

a representation known to t'Hooft [29] in 1973. Now the dependence on $\kappa(\nu)$ cancels exactly in the product of (4.20) and (4.24); the result

$$\Pi(\epsilon) = \frac{\kappa_1}{\epsilon} \exp\left\{-\int\limits_0^{\kappa_1(\epsilon)} \frac{d\lambda}{\lambda + \epsilon f(\lambda)}\right\} \tag{4.25}$$

has none of the apparent g^2 corrections of (4.19). That $\Pi(\epsilon)$ is, moreover, finite as $\epsilon \to 0$ – the exact counterpart of the divergence cancellation observed on the way from (4.18) to (4.19) – follows by writing

$$\frac{1}{\lambda + \epsilon f(\lambda)} = \frac{1}{\lambda + (\epsilon/\beta_0)} + \epsilon\varrho_1(\lambda, \epsilon), \tag{4.26}$$

where $\varrho_1(0,\epsilon)=0$; then

$$\Pi(\epsilon)=\frac{1}{\beta_0}\left[1+\frac{\epsilon}{\beta_0\kappa_1(\epsilon)}\right]^{-1}\exp\left\{-\epsilon\int_0^{\kappa_1(\epsilon)}d\lambda\varrho_1(\lambda,\epsilon)\right\}\,. \tag{4.27}$$

Further subtraction of ϱ_1 at $\lambda=0$ shows that the integral develops also a term $\ln\epsilon$, so finally,

$$\Pi(\epsilon)=\frac{1}{\beta_0}\left[1+O(\epsilon,\epsilon\ln\epsilon)\right]\,. \tag{4.28}$$

This is a compact formulation of the self-consistency mechanism. It clearly shows, through the $\frac{1}{\epsilon}$ factor in (4.14), that the mechanism is *tied to the divergence structure* of the theory, and represents a dynamical exploitation of that structure. In this it is reminiscent, of course, of quantum anomalies, except that here the process seems to have no connection with the breaking of a classical symmetry.

The result is that by imposing the algebraic self-consistency conditions

$$u_{r,2s+1}=-\frac{1}{\beta_0}a_s(u,v,w,x,y,z)\quad(s=0\ldots r) \tag{4.29}$$

in addition to (4.8), the nonperturbative terms of $\Gamma_T^{[r,0]}$ reproduce themselves not only without divergences but also without finite $O(g^2)$ corrections. Note that (4.28) contains only the scheme-independent, leading β-function coefficient, and that it requires evaluation only of the *divergent* parts, a_s, of (4.12/4.13) – the finite parts of those quantities only give terms $O(\epsilon)$ relative to (4.29).

To illustrate these relations, we write conditions (4.29) for the simplest nontrivial, odd-r case of $r=1$, in Landau gauge, and for massless quarks, where eqs. (A.28 – 30) apply with $\hat{m}_F=0$. They are obtained by evaluating the one-loop quantity (4.3) with the $r=1$ vertices and propagators of section 3 and appendix A, using entirely standard techniques of dimensional continuation, Feynman parametrization, and symmetric integration, and by extracting the divergent parts as in eqs. (4.12/4.13). One finds

$$\begin{aligned}\beta_0 u_{1,3} \;=\;& \frac{N_C}{2}\Big\{6\left[u_{1,1}(x_{1,3}+2x'_{1,4})-x_{1,4}\right]\\
&+u_{1,2}\left[+5x_{1,1}-\frac{5}{6}x_{1,3}+\frac{13}{3}x'_{1,2}+6x'_{1,4}-\frac{20}{3}x_{1,6}\right]\Big\}\\
&+\frac{N_C}{2}\left\{\frac{1}{2}(y_{1,4}+y_{1,8})-v_{1,1}(y_{1,2}+y_{1,5}+y_{1,9})\right.\\
&\left.+\frac{1}{2}v_{1,2}(2y_{1,2}+y_{1,5}+y_{1,9})+\frac{1}{6}u_{1,2}y_{1,2}\right\}\\
&-N_F\Big\{w_{1,3}(2z_{1,0}-8z_{1,2}+2z_{1,3})+(3w_{1,1}^2+2w_{1,1}w_{1,2}+w_{1,2}^2)(2z_{1,2}-z_{1,3})\\
&+2(w_{1,2}-w_{1,1})(z_{1,1}+z_{1,2}w_{1,2})+2z_{1,4}+(w_{1,2}+w_{1,1})z_{1,5}+u_{1,2}\left[\frac{1}{3}(2z_{1,0}+z_{1,3})\right]\Big\}\end{aligned} \tag{4.30}$$

$$
\begin{aligned}
\beta_0 u_{1,1} \quad = \quad & \frac{N_C}{2}\left[-11x_{1,1}+6(u_{1,1}-x'_{1,4})+\frac{5}{6}x_{1,3}-\frac{13}{3}x'_{1,2}+\frac{20}{3}x_{1,6}\right] \\
& +\frac{N_C}{2}\left[(v_{1,2}-v_{1,1})+\frac{1}{2}(y_{1,1}+y_{1,3}-\frac{1}{3}y_{1,2})\right] \\
& +\frac{N_C}{2}\left[+\frac{9}{2}u_{1,1}\right] \\
& +N_F\left[-2w_{1,3}+\frac{1}{3}\left(2z_{1,0}+z_{1,3}\right)+2(w_{1,1}-w_{1,2})z_{0,1}-2z_{0,4}-(w_{1,1}+w_{1,2})z_{0,5}\right]
\end{aligned}
\tag{4.31}
$$

plus, of course, the $u'_{1,2}=u_{1,2}$ of eq. (4.8). (An obvious simplified notation has been employed for the fermion-vertex coefficients z of (A.45/46).) Together with analogous conditions for the ghost and quark self-energies, these will form an algebraic system (linear at this level) determining $u_1 \ldots u_3$, and the other self-energy constants, in terms of the nonperturbative coefficients x, y, z of the three-point vertices.

After this extraction of $p=0$ nonperturbative parts, the remainder of the DS dressing term,

$$
\left(\frac{g_0}{4\pi}\right)^2 \left[J_0^{(r)}(k^2) - J_0^{(r)}\left(-\overline{u}\Lambda^2\right)\right] , \tag{4.32}
$$

still has a logarithmic divergence. In this remainder, the quantity (4.15/20) no more occurs (the noninteger power that does occur contains the scale k^2), and treatment of its coupling structure, under boundary conditions (2.3) and (2.5), must therefore revert to the "normal", perturbative pattern. Isolating the divergent piece with $R=MS$ gives,

$$
\left(\frac{g_0}{4\pi}\right)^2 \left[\frac{1}{2}N_C\left(\frac{13}{3}\right)-\frac{2}{3}N_f\right]\left(\frac{1}{\epsilon}\right)\left(k^2+\overline{u}\Lambda^2\right) , \tag{4.33}
$$

which apart from the $\overline{u}$ term is precisely the (Landau- gauge) *perturbative* one-loop divergence. This is as expected, since by the very construction of our vertices, (4.32) must contain the correct perturbative limit. Appealing now once more to our basic condition (2.5) to demand that this divergence should be removable by the perturbative counterterm, one finds that the choice

$$
\overline{u} = 0 \tag{4.34}
$$

is dictated uniquely, eliminating the apparent arbitrariness in the matching condition (4.11). This observation is particularly interesting in the (non-QCD) context of the even-r "particle" sequence with a real-axis propagator pole, since it shows that a nonzero mass term, $u_{r,1}\Lambda^2$, can in principle be generated *while using only the massless perturbative counterterm.*

The finite term of (4.32) in $R=MS$,

$$
\left\{\left[\frac{g(\nu)}{4\pi}\right]^2 \Gamma_{TT}^{[r,1)}(k^2)\right\}_{MS} = \left[\frac{g(\nu)}{4\pi}\right]^2 (k^2)\left\{K_0^{(r)}(k^2)-\left[\frac{1}{2}N_C\left(\frac{13}{3}\right)-\frac{2}{3}N_f\right]\left(\frac{1}{\epsilon}\right)\right\}_{\epsilon\to 0} , \tag{4.35}
$$

where

$$K_0^{(r)}(k^2) = \frac{J_0^{(r)}(k^2) - J_0^{(r)}(0)}{k^2}\,, \tag{4.36}$$

represents the renormalized, one-loop, radiative correction, having errors both of order g^4 in the perturbative and of type $e(r+1)$ in the nonperturbative direction.

A number of comments are in order here:

(i) The divergence content of the one-loop dressing functional is now exhausted. With $[r, 0]$ input, it has just been sufficient for producing the divergence (4.33) in the perturbative correction, curable by perturbative means, and for triggering the "eating" mechanism of eq. (4.28) in the $r+1$ nonperturbative terms, as needed for self-reproduction of the input. It does not, therefore, generate the terms of the next higher order $(r+1)$ automatically, as it does in the perturbative direction. Even less can it define a unique splitting, valid uniformly at all k^2, into order$-g^0$ and order$-g^2$ terms; this can only be approached asymptotically in the $r >> 1$ limit. At finite r, output-input matching can be achieved only in a finite number of comparison data.

(ii) While perturbative renormalization has now gone through completely at one loop, one notes that it has done so with a peculiar new twist: for the vertex function as a whole, the renormalization is no more multiplicative, as in pure perturbation theory, since the nonperturbative terms, by (4.28), have established themselves in a finite manner. At best, one may write the renormalization process as

$$(\Gamma_T)_R = (Z_3)_R\, \Gamma_T^{(\mathrm{pert})} + \left(\Gamma_T - \Gamma_T^{(\mathrm{pert})}\right)\,, \tag{4.37}$$

with Z_3 the perturbative gluon-field renormalization constant, but not as one overall rescaling. (This property must hold already in the OPE if one wants to avoid nonlocal counterterms.)

(iii) The observation that the self-consistency mechanism is tied to the divergences of DS loops immediately implies that nonperturbative terms of the present type cannot establish themselves in superficially *convergent* vertices at one loop. The argument can be extended, however [7]: an n-loop contribution to a superficially convergent vertex has prefactor $(g_0^2)^n$ but can have at most $n-1$ divergent subintegrations, which through eq. (4.28) can "eat" at most $n-1$ powers of g_0^2, so that the result is at best an order-g^2 correction but never of order g^0. This is noteworthy because it shows that, at least for the limited purpose of determining zeroth-order nonperturbative parts, the infinitely coupled nature of the DS system can be beaten without introducing "decoupling" approximations: *the self-consistency problem of* $\Gamma^{[r,0]}$ *is strictly confined to the set of seven DS equations for the superficially divergent vertices.*

(iv) The fact that the one-loop evaluation of Φ has produced both terms of order g^0 and g^2 demonstrates that there is a decoupling of the loop and perturbative orders, l and p – again a feature

presumably present in any truly nonperturbative solution. A discussion of this aspect has been given in the first of refs. [7] and need not be repeated here. Strictly speaking it implies that to fully characterize an approximant $\Gamma^{[r,p]}$, one would need still another index l, of a technical nature, indicating from what loop order $l \geq p$ the r.h. sides of the 0-th order self-consistency equations (which become power series in $\frac{1}{\beta_0}$) and the $1 \leq p' \leq p$ corrections have been determined.

Eqs. (4.30/31) represent, of course, only a small subset of the full set of equations for the seven vertices (1.14). To establish the full set of self-consistency conditions by isolating the divergent parts of all DS interaction terms for these vertices represents, even at the lowest nontrivial ($r = 1$ and $l = 1$) level, a substantial research program, on which work is in progress. The example of (4.30/31) should however be sufficient to demonstrate that those divergent parts are considerably richer in structure than the perturbative ones, and that in this sense the evaluation with nonperturbatively modified diagram elements, $\Gamma^{[r,0]}_{sdiv}$, squeezes more dynamical information from the DS dressing functionals.

5 Special aspects of the quasiparticle subsequence

Among the dynamical possibilities opened up by the extended iterative scheme, the "quasiparticle" subsequence – with r odd and only complex-conjugate singularities in the elementary two-point functions D_T and S_F – clearly is the most interesting one for QCD. Yet its complex singularities seem to raise special questions of interpretation and of possible violation of physical principles. While some of these were mentioned in a purely $r = 1$ context in the second of refs. [7], the crucial ones can be addressed adequately only with the perspective provided by the full sequential approximation. In the following we therefore discuss these conceptual questions one by one under appropriate catchwords.

5.1 Short-lived elementary excitations

The spacetime propagation characteristics implied by the two-point vertices (2.18) and (A.25) of the quasiparticle subsequence are exhibited by continuing the corresponding propagators, $D_T^{[r,0]}$ and $S_F^{[r,0]}$, to Minkowskian (k_M) space and Fourier-transforming to Minkowskian x_M space (we recall that the order of these two steps is essential [30] if one is to arrive at the *time-ordered* Minkowskian two-point functions). We write results only for the scalar function $\tilde{D}_T(x_M^2)$, since the qualitative points will be the same for the two invariant functions of $\tilde{S}_F(x_M)$. Working from (2.23), one has

$$\tilde{D}_T^{[r,0]}\left(x_M^2\right) = \frac{i\Lambda^2}{(2\pi)^2} \sum_{s=0}^{(r-1)/2} 2Re\left\{\left[\varrho_{r,4s+2}\sigma_{r,4s+3}\right] \frac{K_1\left(i\sqrt{\sigma_{r,4s+3}}\Lambda\sqrt{x_M^2}\right)}{i\sqrt{\sigma_{r,4s+3}}\Lambda\sqrt{x_M^2}}\right\}, \tag{5.1}$$

where K_1 is a modified Bessel function. By eq. (2.24), i.e. by virtue of asymptotic freedom, the small-distance behavior of this expression is that of a free propagator. At large timelike distances ($x_M^0 \to \pm\infty$ or $x_M^2 \to +\infty$), assuming that $s = 0$ refers to the pole pair closest to the origin of the k^2 plane, the asymptotic behavior is

$$\tilde{D}_T^{[r,0]}(x_M^2) \to \frac{\chi_{r,0}\Lambda^{\frac{1}{2}}}{\left(2\pi\sqrt{x_M^2}\right)^{\frac{3}{2}}} \, e^{-\gamma_{r,0}\sqrt{x_M^2}} \cos\left(\omega_{r,0}\sqrt{x_M^2} + \varphi_{r,0}\right) \, , \tag{5.2}$$

with real parameters defined by

$$\begin{aligned} \Lambda\sqrt{\sigma_{r,3}} = & \quad \omega_{r,0} - i\gamma_{r,0} = \Lambda\sqrt{\sigma_{r,1}^*} \, , \\ \varrho_{r,2} i^{-\frac{3}{2}} (\sigma_{r,3})^{\frac{1}{4}} = & \quad \chi_{r,0} \, e^{-i\varphi_{r,0}} = \left[\varrho_{r,0} i^{\frac{3}{2}} (\sigma_{r,1})^{\frac{1}{4}}\right]^* \, . \end{aligned} \tag{5.3}$$

In contrast to the "radiative" behavior of a real-mass propagator at timelike distances, this function decays exponentially, with an inverse lifetime, $\tau^{-1} = \gamma_{r,0}$, controlled by the imaginary parts of the leading k^2-plane pole pair. Note that the function decreases in *both* directions of the time axis. It describes an elementary excitation of the (transverse) gluon field, $A_T^\mu(x)|0>$, that can exist only as part of a microscopically short-lived intermediate state.

5.2 Fields without free-field limits

The full Minkowski-space Green functions for N elementary QCD fields, denoted generically φ_i with two-point functions $\tilde{\Delta}_i$, of which at least one is a transverse-gluon field with propagator (5.1), have structure

$$\begin{aligned} \tilde{G}_N(x_1 \ldots x_N) \equiv & \quad < 0|T\left\{\varphi_1(x_1)\ldots A_T(x_k)\ldots\varphi_N(x_N)\right\}|0> \\ = & \quad \tilde{D}_T(x_k - x_l)\tilde{G}_{N-2}(\ldots) + \Big[\tilde{D}_T(x_k - x_l)\tilde{D}_T(x_m - x_n) + \\ & \quad + (\text{permutations})\Big] \; \tilde{G}_{N-4}(\ldots) + \; (\text{other disconnected terms}) \\ & \quad + \int d^4y_1 \ldots \int d^4y_N \tilde{\Delta}_1(x_1 - y_1)\ldots\tilde{D}_T(x_k - y_k)\ldots\tilde{\Delta}_N(x_N - y_N) \\ & \quad \times \tilde{T}_N(y_1 \ldots y_k \ldots y_N) \, , \end{aligned} \tag{5.4}$$

where $\tilde{T}_N$ is the connected and amputated function. For $x_k^0 \to \pm\infty$ with the other x_i fixed, the disconnected-gluon terms, if any, vanish exponentially by (5.2). The behavior of terms with only connected gluons, as exemplified by the fully connected last term of (5.4), is also dominated by relation (5.2) for the $\tilde{D}_T(x_k - y_k)$ factor, provided the y_k integration is sufficiently convergent. The last point involves a subtlety: as discussed in sect. 3 and the appendix, the nonperturbative vertices from which $\tilde{T}_N$ is built have one real-axis pole (at $k_M^2 = u_{r,2}\Lambda^2$) in the squared momentum of each

external gluon, and such poles would seem to cause insufficiently convergent, "radiative" behavior of the $\tilde{T}_N$ in (5.4) with respect to $(y_k)_M$. However, as emphasized in connection with (4.8), the DS self-consistency conditions see to it that propagators acquire zeroes at the positions of vertex poles, so that the unamputated function, with propagators on all legs, retains only the *complex* momentum-space propagator poles, and therefore overall exponential-decay factors as in (5.2) for all transverse-gluon legs. The argument therefore goes through – the connected term, too, vanishes exponentially as $x_k^0 \to \pm\infty$.

Now if all Green functions involving an $A_T(x)$ field vanish in this way, then all matrix elements of $A_T(x)$ between normalizable states decay exponentially as $x^0 \to \pm\infty$: the field has *weak limits of zero* at timelike infinity, in contrast to fields with stable-particle, real-axis propagator poles that are known to have nonvanishing weak limits representing free fields [31].

A different perspective on the same property results from looking at the quantity which "normally" – i.e. for fields with nonvanishing weak free-field limits – defines S-matrix elements: the r.h.s. of the reduction formula. This is obtained from the connected piece of (5.4) by going back to k_M space and conducting a residue search for all squared momenta approaching real mass-shell values:

$$\begin{gathered}(S - \mathbb{1})(k_1, \dots, k_k, \dots, k_N) \propto \\ \lim_{k_1^2 \to \mu_1^2} \dots \lim_{k_k^2 \to \mu_k^2} \dots \lim_{k_N^2 \to \mu_N^2} \Big\{ (k_1^2 - \mu_1^2) \dots (k_k^2 - \mu_k^2) \dots (k_N^2 - \mu_N^2) \times \\ \times \Big[\Delta_1(k_1^2) \dots D_T(k_k^2) \dots \Delta_N(k_N^2) T_N(k_1, \dots k_k, \dots k_N) \Big] \Big\} \ . \end{gathered} \tag{5.5}$$

In the "normal" case, the full propagators Δ_i have particle poles at mass shells $k_i^2 = \mu_i^2$, and the residue search locks in at those points to give a nonzero S-matrix element, proportional to the all-on-shell T. For a gluon with propagator (5.1), $D_T(k_k^2)$ has only complex pole pairs staying at least a distance $(Im\sigma_{r,1})\Lambda^2$ away from the real k_k^2 axis, and by its numerator zero at $k_k^2 = u_{r,2}\Lambda^2$ cancels the real-axis pole of T. The residue search at real k_k^2, no matter what the value of the real μ_k^2, therefore gives zero. (The longitudinal gluons do retain their real-axis poles at $k^2 = 0$, as do ghosts in the present scheme, but these unphysical modes are of course excluded from the S matrix by appropriate projectors not explicitly indicated in (5.5).) One concludes that S-matrix elements involving at least one external gluon (or, by extension, quark) are, depending on semantic taste, either nonexistent (if one considers them as defined only for fields with nonzero free-field limits) or zero (if one applies definition (5.5) in an extended sense). The physical implication is the same - gluons and quarks are not asymptotically detectable reaction products.

5.3 Violation of Causality?

To avoid confusion here, we must recall that arguments deducing absence of complex singularities in amplitudes from microcausality refer to *S-matrix elements* (the propagators themselves, like all off-

shell Green functions, are always "acausal" by construction, i.e. nonvanishing at spacelike separations) and to those of the *elementary* fields (composite hadronic fields, being nonlocal, have no strictly microcausal commutators). The question then arises whether one-particle reducible contributions to S-matrix elements, in which one of the external sums-of-momenta flows through a single gluon or quark propagator, violate causality through their complex poles. (The simplest amplitudes of this type are the tree graphs of fig. 3). The answer is no, since these amplitudes also have external legs corresponding to the elementary (gluon and quark) fields, and therefore are zero (or must be declared nonexistent) as discussed in 5.2 above. In other words, the *same* set of propagators that seem to raise causality problems when occurring on internal lines, also ensure the vanishing of those amplitudes *taken as S-matrix elements* by being present on the external lines.

5.4 Reflection Positivity?

In Euclidean-coordinate (x_E) space, the propagator $\tilde{D}_T^{[r,0]}(x_E^2)$ has an asymptotic form similar to (5.2) but with the roles of $\omega_{r,0}$ and $\gamma_{r,0}$ interchanged. At $r = 1$ and generally for low r in the quasiparticle sequence, such a propagator is not a purely positive function of x_E^2, as required by reflection positivity. Its graph, shown schematically in fig. 4, has negative "overshoots" caused by the oscillations of K_1 Bessel functions at complex arguments. This problem is potentially serious, but not inevitably so. It is quite possible, and indeed likely (given the fact that the DS equations are compatible with physical requirements like positivity) that it gets cured gradually, by the coherent admixture of faster oscillations from more distant complex-pole pairs, as the order r is increased. (This in fact is the normal situation in any discrete Fourier approximation of a positive function.) If this occurs, then the lack of reflection positivity at low r should simply be viewed as one of the natural errors expected in the low orders of any approximation scheme. It will basically be no more serious than e.g. the well-known violation of unitarity by the Born approximation, i.e. by low-order perturbation theory. However, it is true that such a conjecture can be checked only by an actual study of the $r \geq 3$ levels.

If present, restoration-by-interference of reflection positivity would highlight a deeper point about the short-lived elementary excitations: the single complex-pole pair of the $r = 1$ propagator makes sense, strictly speaking, only as the lowest level of an improving sequence of approximations. While a stable-particle pole makes physical sense in isolation (describing then a free field), a quasiparticle pole pair can support itself only through the presence of interactions, which manifest themselves as dressing cuts. It is tempting, though at present speculative, to view this in parallel with the fact that a nonabelian gauge field cannot be defined without self-interaction. The formation of short-lived elementary excitations may be a dynamical specialty of nonabelian gauge theories.

5.5 Asymptotic Incompleteness

The short-lived gluon or quark excitations as described by a two-point function in the "quasiqarticle" subsequence are distinct from ordinary decaying particles or resonances. The difference can be made precise by looking at the singularity structure of the propagator: for a resonance, this is characterized by a physical, gauge-fixing independent cut on the real axis, whose discontinuity shows some local enhancement around the resonance energy. This indicates that the resonance will decay into stable, asymptotically detectable fragments, which together carry the open quantum numbers of that resonance. Therefore, it can in principle be recreated by macroscopically preparing beams of those fragments and colliding them. The propagator of the short-lived QCD excitation has no such decay cut. (Again, there is a subtlety – these propagators do have some real-axis cuts, arising from virtual conversion into longitudinal gluons and ghosts, the unphysical degrees of freedom, but these are gauge-fixing artefacts recognizable by their ξ dependence.) The excitation therefore cannot decay into asymptotically detectable configurations carrying its open quantum numbers, and conversely *cannot be prepared in isolation* by colliding macroscopically preparable fragments.

What can be produced from asymptotically controlled initial states are two-gluon and multi-gluon (or multi-quark) configurations compatible with the quantum numbers of the production channel, whose propagation in lowest order is described by product propagators of type

$$\{D(k_1)D(k_2)\dots D(k_n)\}^a \, , \tag{5.6}$$

the notation indicating a coupling to the quantum numbers a. These propagators see to it that the configuration propagates only for a limited time of order Λ^{-1} before the dynamics of the theory forces it to pass its conserved quantities, and share of probability flux, back to some asymptotically accessible exit channels.

The nature of the state space of such a system can at present be inferred only by educated guess. (Although the state space is in principle determined by the full set of Green functions, the prescription to construct the superficially convergent functions by skeleton expansions is far too inexplicit to actually provide a handle for an application of the reconstruction theorems.) Fig. 5 is a schematic representation of such a guess. It depicts the passing of unit total probability through the continuum space $\mathcal{H}_A$ of an ordinary scattering system, where it suffers some delay called a resonance, and through a subspace of exactly conserved quantum numbers, having an asymptotically accessible portion $\mathcal{H}_A$, in QCD. In the latter case, probability flux can partially and temporarily – for times of order Λ^{-1} – leak out of $\mathcal{H}_A$ and populate a "closed", not directly accessible space, $\mathcal{H}_C$. The presence of the latter implies that $\mathcal{H}_A$ does not, together with the bound-state space $\mathcal{H}_B$, exhaust the full state space, i.e. that *such a system lacks asymptotic completeness*, a feature unfamiliar but by no means unacceptable physically. (Since temporal evolution in such a system maps a smaller onto a larger space and vice

versa, it may have to be described by operators that are isometric but not unitary, a feature not excluded by the requirement of probability conservation.)

It is natural to inquire about the evolution of a (purely hypothetical, because of the impossibility of macroscopic preparation) single gluonic excitation, which would live in a subspace of conserved quantum numbers of the pure $\mathcal{H}_C$ type, without any $\mathcal{H}_A$ and $\mathcal{H}_B$ portions. This evolution would exhibit some analogy with that of a free particle in quantum mechanics, whose wave function, as long as no position measurement is performed, spreads indefinitely over all space. In a similar manner, the single short-lived excitation would convert into more and more complex multiexcitations, all short-lived, and thereby spread indefinitely over the whole $\mathcal{H}_C$ space.

5.6 S-Matrix Unitarity

While evolution projected onto the asymptotically accessible subspace $\mathcal{H}_A$ is not unitary at small times, due to the partial leakout of probability into the $\mathcal{H}_C$ space, the fact that the leaking is dynamically limited to times of order Λ^{-1} guarantees that between *asymptotic* times, $t = -\infty$ and $t' = +\infty$, probability conservation does hold on that subspace alone, since all probability flux fed into the system through scattering eventually gets pushed back into asymptotically accessible exit channels. In other words, one expects the S matrix to remain strictly unitary.

It is instructive to check this expectation out in a more formal way by asking whether the relation

$$Im < b|T|a >= -\sum_n < b|T^\dagger|n >< n|T|a > \tag{5.7}$$

is compatible with the presence of the "closed" subspace $\mathcal{H}_C$. If at least one of the two states $|a >, |b >$ is in that subspace, then by the discussion around eq. (5.5), the l.h.s. and at least one of the factors on the r.h.s. must be set equal to zero, and the relation is trivially fulfilled. If both $|a >$ and $|b >$ are in the asymptotically accessible space $\mathcal{H}_A$, then states $|n >$ in $\mathcal{H}_C$, again by the arguments of sect. 5.2, make no contribution to the sum on the r.h.s. But then the imaginary part on the l.h.s., which is equal to the discontinuity across the timelike real axis of the relevant energy-squared variable for the $a \to b$ process, does not receive contributions from such states either: intermediate states in $\mathcal{H}_C$ are described by propagators of type (5.6), and integration over them produces *pairs* of cuts at complex-conjugate locations, which together always leave the amplitude continuous and real along the real s axis (even though some of them do start at points on that axis), i. e. which do not produce physical absorptive parts. (This can be checked out in detail for e. g. the one-loop terms of fig. 2 with the $r = 1$ propagators). One concludes that it is indeed consistent for eq. (5.7) to hold on the asymptotically accessible space alone.

The discussion of this section may be summarized by stating that, perhaps contrary to first appearances, a solution $\{\Gamma^{[r,p]}|r \text{ odd}\}$ of the quasiparticle type with complex-conjugate propagator sin-

gularities does not necessarily imply violation of physical principles beyond the natural errors expected in the lower orders of any systematic approximation scheme. Such a solution is in fact well suited for a description of the elementary QCD excitations where they come closest to being "seen", namely at the origins of jet events.

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Appendix

A Other Superficially Divergent Vertices

A.1 Four-Vector Vertex

This is the highest of the superficially divergent vertices (1.14). Its complexity with respect to color and Lorentz structure exceeds by an order of magnitude that of all the other Γ_{sdiv} combined, and is the main source of technical difficulty in the $p = 0$ self-consistency problem. The number of color-basis tensors $C^{(i)}$ – each a fourth-rank object over the adjoint representation of $SU(N_C)$ – in the decomposition

$$\Gamma^{\kappa\lambda\mu\nu}_{4V,abcd} = \sum_i C^{(i)}_{abcd}\ \Gamma^{\kappa\lambda\mu\nu}_{4V(i)} \tag{A.1}$$

is eight for $N_C = 3$ and nine for $N_C \geq 4$. For brevity, we refer the reader to ref. [11] for the general case and write an example of a suitable color basis only for the case of immediate interest in QCD, viz. $N_C = 3$:

$$\begin{aligned} C^{(1)}_{abcd} &= \delta_{ab}\delta_{cd}\,,\ C^{(2)}_{abcd} = \delta_{ac}\delta_{bd}\,,\ C^{(3)}_{abcd} = \delta_{ad}\delta_{bc}\,, \\ C^{(4)}_{abcd} &= f_{abn}f_{cdn}\,,\ \left[C^{(5)} - C^{(6)}\right]_{abcd} = f_{acn}f_{dbn} - f_{adn}f_{bcn}\,, \\ C^{(7)}_{abcd} &= d_{abn}f_{cdn}\,,\ C^{(8)}_{abcd} = d_{acn}f_{dbn}\,,\ C^{(9)}_{abcd} = d_{adn}f_{bcn}\,. \end{aligned} \tag{A.2}$$

The combination $C^{(5)} + C^{(6)}$ is not listed, as it equals $-C^{(4)}$ by virtue of the Jacobi identity. In the many situations where minimality of the basis is not crucial but manifest Bose and crossing symmetry are, we shall nevertheless use the nine-dimensional set including $C^{(5)}$ and $C^{(6)}$ separately.

For Lorentz structure, we again concentrate on the totally transverse pieces defined in analogy with (3.2), which have tensor decompositions

$$\begin{aligned} \Gamma^{\kappa'\lambda'\mu'\nu'}_{4T(i)}(p_1\ldots p_4) = \quad & t^{\kappa'\kappa}(p_1)t^{\lambda'\lambda}(p_2)t^{\mu'\mu}(p_3)t^{\nu'\nu}(p_4) \\ & \times \sum_{k=0}^{2}\sum_j L^{\kappa\lambda\mu\nu}_{(k,j)}(p_1\ldots p_4)\ G_{(i)k,j}\left(p_1^2\ldots p_4^2; s,t,u\right)\,, \end{aligned} \tag{A.3}$$

with $\sum p_i = 0$ again understood. Linearly independent Lorentz tensors $L_{(k,j)}$, constructed from the Euclidean metric $\delta^{\alpha\beta}$ and three independent combinations (denoted generically $k_{1,2,3}$) of the p_i, come with three mass dimensions, $2k = 0, 2$, and 4. They include three dimensionless ($k = 0$) tensors,

$$L^{\kappa\lambda\mu\nu}_{(0,1)} = \delta^{\kappa\lambda}\delta^{\mu\nu}\,,\ L^{\kappa\lambda\mu\nu}_{(0,2)} = \delta^{\kappa\mu}\delta^{\lambda\nu}\,,\ L^{\kappa\lambda\mu\nu}_{(0,3)} = \delta^{\kappa\nu}\delta^{\lambda\mu}\,, \tag{A.4}$$

plus 54 independent $2k = 2$ tensors $L_{(1,j)}$ of type $\delta^{\alpha\beta}k^{\gamma}_m k^{\delta}_n \quad (m, n \in \{1, 2, 3\})$, of which 24 contribute to the totally transverse vertex, plus 81 independent $2k = 4$ tensors $L_{(2,j)}$ of type $k^{\kappa}_k k^{\lambda}_l k^{\mu}_m k^{\nu}_n \quad (k, l, m, n \in$

$\{1,2,3\}$), of which only 16 contribute to the $4T$ vertex. The perturbative zeroth-order vertex contains only the $k=0$ Lorentz tensors (A.4) and has the expressions

$$\begin{aligned}\Gamma^{(0)\mathrm{pert}}_{4V} =& C^{(4)}\left[L_{(0,2)}-L_{(0,3)}\right]+C^{(5)}\left[L_{(0,3)}-L_{(0,1)}\right]+C^{(6)}\left[L_{(0,1)}-L_{(0,2)}\right] \\ =& C^{(4)}\left[\tfrac{3}{2}\left(L_{(0,2)}-L_{(0,3)}\right)\right]+\tfrac{1}{2}\left(C^{(5)}-C^{(6)}\right)\left[-2L_{(0,1)}+L_{(0,2)}+L_{(0,3)}\right]\end{aligned} \tag{A.5}$$

in the linearly dependent but crossing symmetric and in the minimal color basis (A.2), respectively. We do not write full enumerations of the $k>0$ objects here, which will be of interest only in connection with detailed loop computations. For a first exploration of the solution described here, it will be reasonable in any case to start by looking for approximate self-consistency with a strongly restricted tensor structure.

One would expect to treat the invariant functions $G_{(i)k,j}$ associated with these tensors as functions of six independent Lorentz-invariant variables, an example being the six scalar products formed from the three conserved total four-momenta in the three crossed two-gluon channels,

$$\begin{aligned}P &= p_1+p_2 = -(p_3+p_4)\,, \\ R &= p_1+p_3 = -(p_2+p_4)\,, \\ Q &= p_1+p_4 = -(p_2+p_3)\,.\end{aligned} \tag{A.6}$$

Again, however, it turns out that one has no freedom here: dynamical consistency in the three-gluon DS equation connecting Γ_{3V} to Γ_{4V}, together with crossing symmetry, dictates that rational approximation of $G's$ be performed with respect to the *seven* variables indicated in (A.3), where

$$s=P^2\,, \qquad u=R^2\,, \qquad t=Q^2\,, \tag{A.7}$$

and with the relation expressing their linear dependence,

$$s+u+t=\sum_{i=1}^{4}p_i^2\,, \tag{A.8}$$

being carried as a subsidiary condition. This would seem to considerably complicate the writing and proper restriction of FDRA's were it not for two simplifying features which we anticipate: the various additive terms of (A.1/A.3) can be grouped as either crossing triplets or crossing singlets, and if one restricts attention to $\Gamma^{[r,0]}_{4T}$, the nonperturbatively modified vertex of zeroth perturbative order, then self-consistency is possible for a simplified structure, where in each member of a crossing triplet the invariant-function *denominator* contains only one of the Mandelstam variables s,t,u at a time, while for a crossing singlet it does not depend on them at all. Moreover, pole positions in FDRA's with respect to the four p_i^2 and with respect to the variables (A.7), which a priori could be chosen differently,

are forced by DS consistency to be in fact the same. Thus the invariant function for the s-channel member of a crossing triplet would possess r-th degree FDRA's of the form

$$G^{[r,0]}_{(i)k,j} = \frac{N^{(r)}_{4V(i)k,j}(p_1^2 \dots p_4^2; s; \frac{1}{2}(u-t))}{\left[\prod_{\sigma=1}^{r}\left(s + u''_{r,2\sigma}\Lambda^2\right)\right]\left\{\prod_{l=1}^{4}\left[\prod_{\sigma=1}^{r}\left(p_l^2 + u''_{r,2\sigma}\Lambda^2\right)\right]\right\}}\,, \tag{A.9}$$

and the other two members of the triplet would follow from this by the two crossing operations

$$\begin{aligned} (b,\lambda,p_2) &\longleftrightarrow (c,\mu,p_3); \quad P \leftrightarrow R, s \leftrightarrow u\,; \\ (b,\lambda,p_2) &\longleftrightarrow (d,\nu,p_4); \quad P \leftrightarrow Q, s \leftrightarrow t\,. \end{aligned} \tag{A.10}$$

The mass dimension $-2k$, as well as the various restrictions and Bose-symmetry properties, are built into the numerator polynomial,

$$\begin{aligned} N^{(r)}_{4V(i)k,j}\left(p_1^2 \dots p_4^2; s; \tfrac{1}{2}(u-t)\right) = &\quad (\delta_{i4}+\delta_{i5}+\delta_{i6})\,\delta_{k0}\left(s p_1^2 p_2^2 p_3^2 p_4^2\right)^r \\ + \sum_{m_1 \dots n_2 \geq 0} C^{(i)k,j(r)}_{m_1 m_2 m_3 m_4 n_1 n_2} &\left[\prod_{l=1}^{4}(p_l^2)^{m_l}\; s^{n_1}\left(\tfrac{u-t}{2}\right)^{n_2}\right](\Lambda^2)^{5r-k-(m_1+\ldots+n_2)}\,. \end{aligned} \tag{A.11}$$

Condition (2.3) allows nonzero coefficients $C_{m_1 \dots n_2}$ only for

$$m_1 + m_2 + m_3 + m_4 + n_1 + n_2 \leq 5r - k - 1\,. \tag{A.12}$$

The additional restrictions arising from condition (2.5) are now more involved: when Γ_{4V} appears in a 1PI diagram, up to *two* of its four legs may be external. To avoid the occurrence of ultraviolet divergences stronger than the perturbative ones, no term of the vertex should then behave worse than a constant at large loop momenta when (i) any three of its momenta p_i are running in loops and (ii) any two of its momenta are running in a loop. Considering only the case where the $2k$ momenta supplied (for $k > 0$) by the $L_{(k,j)}$ tensor are all running along in those loops, we find that (i) imposes the four restrictions

$$\left[\sum_{l=1}^{4}(1-\delta_{li})m_l\right] + (n_1+n_2) \leq 4r - k \qquad (i = 1,2,3,4)\,, \tag{A.13}$$

whereas (ii) implies, for the function (A.11), the six restrictions

$$\begin{aligned} (m_i + m_j) + (n_1 + n_2) &\leq 3r - k \\ (i,j) &= (1,3),(1,4),(2,3),(2,4)\,; \end{aligned} \tag{A.14}$$

$$\begin{aligned} 2(m_1+m_2) + n_2 &\leq 4r - 2k\,; \\ 2(m_3+m_4) + n_2 &\leq 4r - 2k\,. \end{aligned} \tag{A.15}$$

In writing (A.15) we have used the fact that

$$u - t = (p_2 - p_1) \cdot (p_4 - p_3)\,. \tag{A.16}$$

In other cases where some of the "tensorial" momenta stay out of the loops, these restrictions may be relaxed slightly, but the additional freedom gained in this way is minor.

For the self-consistency problem, the relevant structural property emerging from the above is that with respect to the variables (A.7), the $p = 0$ vertex may be assumed to have a simplified partial-fraction decomposition which is the analog of (3.10),

$$\begin{aligned} \Gamma_{4T}^{[r,0]} = \quad & E_0^{(r)} + \sum_{n=1}^{r} \left[E_{n(s)}^{(r)} \left(\frac{\Lambda^2}{s + u''_{r,2n}\Lambda^2} \right) + \right. \\ & \left. + E_{n(u)}^{(r)} \left(\frac{\Lambda^2}{u + u''_{r,2n}\Lambda^2} \right) + E_{n(t)}^{(r)} \left(\frac{\Lambda^2}{t + u''_{r,2n}\Lambda^2} \right) \right]. \end{aligned} \tag{A.17}$$

Here $E_{n(u)}, E_{n(t)}$ are the crossing partners of $E_{n(s)}$ in the sense of (A.10), and each of the E_n tensors in turn has a partial-fraction decomposition with respect to the p_i^2 variables whose form is exemplified by

$$E_{n(s)}^{(r)} = \sum_{\alpha,\beta=0}^{1} \sum_{\sigma=\alpha}^{\alpha r} \sum_{\tau=\beta}^{\beta r} E_{n(s)\sigma\tau} \left(\frac{\Lambda^2}{p_1^2 + u''_{r,2\sigma}\Lambda^2} \right)^{\alpha} \left(\frac{\Lambda^2}{p_2^2 + u''_{r,2\tau}\Lambda^2} \right)^{\beta}. \tag{A.18}$$

Here the $E_{n(s)\sigma\tau}$ functions still have dependence on p_3^2, p_4^2.

A.2 Ghost vertices

The well-known fact that diagrams, whether perturbative or dressed, with external G and $\overline{G}$ lines factor out one power of an external momentum per $G\overline{G}$ pair, causes the Dyson-Schwinger equation (1.13) for the inverse ghost propagator to assume the special form

$$\delta_{ab}\Gamma_{G\overline{G}}(p^2) = -\delta_{ab}p^2 \left\{ 1 + \left(\frac{g_0}{4\pi} \right)^2 \tilde{I}\left[\tilde{D}, D, \Gamma_{GV\overline{G}} \right](p^2) \right\}, \tag{A.19}$$

where the dimensionless, logarithmically divergent loop integral $\tilde{I}(p^2)$ involves the $\Gamma_{GV\overline{G}}$ three-point vertex. Rational approximants of the odd-r sequence for the ghost propagator $\tilde{D}$ are therefore of a type we omitted as exotic when discussing D_T – they generically have two different real-axis poles, one massless and one massive, which for an unphysical excitation is not impossible. For example, the counterpart of (2.15) reads,

$$\tilde{D}^{[1,0]}(p^2, \Lambda^2) = \frac{p^2 + v_{1,2}\Lambda^2}{p^2(p^2 + v_{1,1}\Lambda^2)}\,. \tag{A.20}$$

This includes the special cases $v_{1,2} = v_{1,1}$, where the $r = 1$ function reduces to an $r = 0$ function with one massless real-axis pole, $v_{1,2} = 0$, where the same happens but with a mass (generally gauge-fixing

dependent) for the FP ghost, and $v_{1,1} = 0$, where the ghost develops a second-order pole at $p^2 = 0$, as it does under the Gribov-Zwanziger mechanism [14,12]. For a generic gauge fixing, however, (A.20) and its higher-r extensions are likely to remain the relevant forms.

In the $\Gamma_{GV\overline{G}}$ three-point vertex, FDRA's for the invariant functions $\tilde{F}_{(c)i}(c = f$ or $d)$ defined by the tensor decomposition

$$\Gamma^{\ \mu}_{abc}(p_1, k, -p_2)_{GV\overline{G}} = f_{abc}\left[p_1^\mu \tilde{F}_{(f)0} + k^\mu \tilde{F}_{(f)1}\right] + d_{abc}\left[p_1^\mu \tilde{F}_{(d)0} + k^\mu \tilde{F}_{(d)1}\right] , \tag{A.21}$$

(where now $p_1 + k - p_2 = 0$), are somewhat more complicated than (3.6) because of the absence of symmetry properties. In particular, denominator zeroes in the ghost-line variables p_1^2 and p_2^2, which we denote as $-\ v'_{r,2s}\Lambda^2$, may differ from those in the gluon variable k^2 even at $r = 1$. The general structure is now

$$\tilde{F}^{[r,0]}_{(c)i} = \frac{N^{(r)}_{GV\overline{G}(c)i}(p_1^2,k^2,p_2^2)}{\left[\prod\limits_{s=1}^{r}\left(p_1^2+v'_{r,2s}\Lambda^2\right)\right]\left[\prod\limits_{s=1}^{r}\left(k^2+\tilde{u}'_{r,2s}\Lambda^2\right)\right]\left[\prod\limits_{s=1}^{r}\left(p_2^2+v'_{r,2s}\Lambda^2\right)\right]} \tag{A.22}$$
$$(c = f \text{ or } d, \quad i = 0 \text{ or } 1, \quad r = 1, 3, 5\ldots) ,$$

where numerator polynomials must respect the perturbative limit

$$\tilde{F}^{(0)\text{pert}}_{(c)i} = \delta_{cf}\delta_{i0} ,$$

as well as restrictions analogous to the $k = 0$ version of (3.8/3.9), but no symmetry restrictions. The explicit form of the $\tilde{F}_{(f)0}$ function for $r = 1$ will again be given for illustration:

$$\begin{aligned}\tilde{F}^{[1,0]}_{(f)0}\left(p_1^2, k^2, p_2^2\right) = \ & 1 + y_1\tilde{\Pi}_{p_1} + y_3\tilde{\Pi}_{p_2} + \\ & + \left(y_6 + y_7\tfrac{k^2}{\Lambda^2}\right)\tilde{\Pi}_{p_1}\tilde{\Pi}_{p_2} + \Pi_k\left[y_2 + \left(y_4 + y_5\tfrac{p_1^2}{\Lambda^2}\right)\tilde{\Pi}_{p_2}\right. \\ & \left. + \left(y_8 + y_9\tfrac{p_2^2}{\Lambda^2}\right)\tilde{\Pi}_{p_1} + y_{10}\tilde{\Pi}_{p_1}\tilde{\Pi}_{p_2}\right]\end{aligned} \tag{A.23}$$

Here $\tilde{\Pi}_p = \Lambda^2/\left(p^2 + v'_2\Lambda^2\right)$ denotes a rational building block analogous to the Π of eq. (3.11). The important point for the DS self-consistency problem is again the existence of a "regular-plus-singular" decomposition, analogous to (3.10), in any one variable; e.g.,

$$\begin{aligned}\left[\Gamma^{\ \mu}_{GV\overline{G}(c)}(p_1, k, -p_2)\right]^{[r,0]} = \ & \tilde{B}^\mu_{(c)0,r}\left(p_1^2, p_2^2; k^2\right) \\ & + \sum_{s=1}^{r}\tilde{B}^\mu_{(c)s,r}\left(p_1^2, p_2^2\right)\left(\frac{\Lambda^2}{k^2+\tilde{u}'_{r,2s}\Lambda^2}\right) ,\end{aligned} \tag{A.24}$$

In refs [7], using heuristically what we would now call the $r = 1$ degree of rational approximation, treatment of the two ghost vertices was subject to the prejudice, taken over from older work of Eichten and Feinberg [9], that vertices do not develop nonperturbative corrections with respect to the momenta of unphysical external lines. This, in particular, kept the ghost propagator $\tilde{D}^{[1,0]}$ in its perturbative form, $1/p^2$. From the more systematic viewpoint advanced here, this prejudice is seen

to be unjustified. Even if DS dynamics would make the case $v_{r,2} = v_{r,1}$ prevail for all r, the function $\tilde{D}^{[r,0]}$, which in contrast to the longitudinal gluon propagator D_L is not protected by an ST identity, would pick up pole-zero pairs at $r \geq 3$ to represent its dressing cuts, and these in turn would demand p_1^2 and p_2^2 denominator factors in (A.22). This is in accord with the fact, emphasized by Lavelle and Schaden [24] in an OPE context, that nonperturbative ghost-antighost vacuum condensates of zeroth perturbative order do exist.

A.3 Fermion propagators.

We use Euclidean γ matrices obeying $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$, so that $\not{p}\not{p} = -p^2$. The flavor-F, inverse propagator of zeroth perturbative order, analogous to (2.18), has odd-r approximants

$$- \Gamma^{[r,0]}_{F\overline{F}}(\not{p}, \hat{m}_F, \Lambda) = \not{p} + \kappa^{(F)}_{r,1} + \frac{\left(\kappa^{(F)}_{r,3}\right)^2}{\not{p} + \kappa^{(F)}_{r,2}} + \sum_{s=2}^{(r+1)/2} \left[\frac{\left(\kappa^{(F)}_{r,2s+1}\right)^2}{\not{p} + \kappa^{(F)}_{r,2s}} + \frac{\left(\kappa^{*(F)}_{r,2s+1}\right)^2}{\not{p} + \kappa^{(F)*}_{r,2s}} \right] . \tag{A.25}$$

Here we encounter a complicating feature: the nonperturbative mass scales $\kappa_{r,i}$ cannot, except in special cases, be written simply as numerical multiples of Λ, since there now exist additional RG-invariant mass scales. These are the extraneous, or Lagrangian, mass scales $\hat{m}_F$, one for each flavor F, that are connected to renormalized quark masses $m_F(\nu)$ through

$$(\hat{m}_F)_R = [m_F(\nu)]_R \exp\left\{ - \int^{g(\nu)} dg' \left[\frac{\gamma_m(g')}{\beta(g')} \right]_R \right\} \tag{A.26}$$

in the scheme indicated by R. The rational character of approximants with respect to Λ then allows fermionic mass scales, such as the κ's and κ^2's of (A.25), to be polynomials in Λ and $\hat{m}_f$. The only restrictions come from condition (2.3), which demands that the bare vertex,

$$- \Gamma^{(0)\mathrm{pert}}_{F\overline{F}} = \not{p} + \hat{m}_F \, , \tag{A.27}$$

be approached in the "perturbative" limit, $\Lambda \to 0$. In general, we then have

$$\kappa^{(F)}_{r,1} = \hat{m}_f + w^{(F)}_{r,1} \Lambda \, , \tag{A.28}$$

$$\kappa^{(F)}_{r,2s} = w'^{(F)}_{r,2s} \hat{m}_f + w^{(F)}_{r,2s} \Lambda \, , \tag{A.29}$$

$$\left(\kappa^{(F)}_{r,2s+1}\right)^2 = \Lambda \left[w'^{(F)}_{r,2s+1} \hat{m}_f + w^{(F)}_{r,2s+1} \Lambda \right] . \tag{A.30}$$

The primed coefficients, w', are absent in the strict chiral limit, $\hat{m}_F = 0$. In the opposite limit, $\hat{m}_F >> \Lambda$, the pole terms of (A. 25) are of order $\Lambda / \hat{m}_f$ relative to the leading $\hat{m}_f$ term of (A. 27). To

illustrate (A.25), we write the analogs of (2.13) and (2.15) for $r = 1$ (omitting now F on mass scales):

$$\begin{aligned} -\Gamma^{[1,0]}_{F\overline{F}} &= \not{p} + \kappa_{1,1} + \frac{\kappa^2_{1,3}}{\not{p}+\kappa_{1,2}} \\ &= \not{p}\left[1 - \frac{\kappa^2_{1,3}}{p^2+\kappa^2_{1,2}}\right] + 1\!\mathrm{I}\left[\kappa_{1,1} + \frac{\kappa_{1,2}\kappa^2_{1,3}}{p^2+\kappa^2_{1,2}}\right] , \end{aligned} \tag{A.31}$$

$$\begin{aligned} S^{[1,0]}_F &= \frac{\not{p}+\kappa_{1,2}}{(\not{p}+\kappa_{1,+})(\not{p}+\kappa_{1,-})} \\ &= \frac{-\not{p}\left[p^2+(\kappa_{1,2}(\kappa_{1,+}+\kappa_{1,-})-|\kappa_{1,\pm}|^2)\right]+1\!\mathrm{I}\left[(\kappa_{1,+}+\kappa_{1,-}-\kappa_{1,2})p^2+\kappa_{1,2}|\kappa_{1,\pm}|^2\right]}{(p^2+\kappa^2_{1,+})(p^2+\kappa^2_{1,-})} , \end{aligned} \tag{A.32}$$

where $1\!\mathrm{I}$ denotes a unit Dirac matrix, and where

$$\kappa^2_{1,\pm} = \left[\frac{1}{2}\left(\kappa^2_{1,1} + \kappa^2_{1,2}\right) - \kappa^2_{1,3}\right] \pm i\left(\kappa_{1,1} + \kappa_{1,2}\right)\sqrt{\kappa^2_{1,3} - \left[\frac{1}{2}\left(\kappa_{1,1} - \kappa_{1,2}\right)\right]^2} . \tag{A.33}$$

A complex-conjugate pair of poles, and therefore a short-lived quark-like excitation, is present if

$$\kappa_{1,1} + \kappa_{1,2} \neq 0 \ \text{ and } \ \kappa^2_{1,3} > \left[\frac{1}{2}\left(\kappa_{1,1} - \kappa_{1,2}\right)\right]^2 . \tag{A.34}$$

Again, for the interpretation to hold at $r \geq 3$ requires a definite pattern of one nearest-to-origin pole pair, plus well separated zero-pole strings identifiable as complex cuts, to emerge as r is increased.

It is interesting to note the tight correlation present in (A.31) between spontaneous chiral-symmetry breaking and the emergence of short-lived quarklike excitations. In the chiral limit $\hat{m}_f = 0$, where the $\kappa_{1,i}$ scales are purely spontaneous (multiples of Λ), chiral-symmetry breaking can be avoided only by having the invariant function associated with the Dirac $1\!\mathrm{l}$ in (A.31) vanish identically, which implies either $\kappa_{1,1} = \kappa_{1,2} = 0$, or $\kappa_{1,1} = \kappa^2_{1,3} = 0$. In both cases, the pole positions (A.33) turn purely real. In other words, whenever short-lived quarks are formed dynamically in the $\hat{m}_f = 0$ case, chiral symmetry must be broken.

The second line of (A.31) moreover illustrates a technical but important point: residues of the two invariant functions of $S^{[1,0]-1}_F$ at the commmon $p^2 = -\kappa^2_{1,2}$ pole cannot be chosen independently but must be related in such a way as to combine into a single pole in the matrix-valued variable $\not{p}$,

$$\frac{-\not{p} + \kappa_{1,2}}{p^2 + \kappa^2_{1,2}} = \frac{1}{\not{p} + \kappa_{1,2}} . \tag{A.35}$$

The reason is that by using unrelated residues one obtains, in general, a propagator S_F with *three* poles, and therefore ends up in the subsequence describing stable quark particles that we do not wish to study here.

A.4 Fermion-vector three-point vertices.

Color structure is strictly identical to that of the bare vertices:

$$\Gamma^{\mu,a}_{F,c,\alpha;\overline{F},c',\beta} = \left(\frac{1}{2}\lambda^a\right)_{cc'} \Gamma^{\mu}_{F,\alpha;\overline{F},\beta}(p_1,k,-p_2) \tag{A.36}$$

(c, c', triplet-color indices; α, β, Dirac-Spinor indices, k, gluon four-momentum, and $p_1 + k - p_2 = 0$). Lorentz (matrix-vector) structure of the $\Gamma^{\mu}_{\alpha\beta}$ amplitudes is then the same as for the QED vertex [32]. We therefore mention only that a tensor decomposition suitable for our purpose,

$$\Gamma^{\mu}_{\alpha\beta}(p_1,k,-p_2) = \sum_{i=1}^{12} W_i\left(p_1^2, k^2, p_2^2\right)(V_i^{\mu})_{\alpha\beta} \,, \tag{A.37}$$

is possible in terms of the twelve matrix-valued vectors,

$$V_1 = \gamma^{\mu}, \quad V_2 = \not{p}_1\gamma^{\mu}, \quad V_3 = \gamma^{\mu}\not{p}_2, \quad V_4 = \not{p}_1\gamma^{\mu}\not{p}_2 \,, \tag{A.38}$$

$$V_5 = r^{\mu}1\!\mathrm{I}, \quad V_6 = \not{p}_1 r^{\mu}, \quad V_7 = r^{\mu}\not{p}_2, \quad V_8 = \not{p}_1 r^{\mu}\not{p}_2 \,, \tag{A.39}$$

$$V_9 = k^{\mu}1\!\mathrm{I}, \quad V_{10} = \not{p}_1 k^{\mu}, \quad V_{11} = k^{\mu}\not{p}_2, \quad V_{12} = \not{p}_1 k^{\mu}\not{p}_2 \,, \tag{A.40}$$

where

$$r^{\mu} = \frac{1}{2}(p_2 + p_1)^{\mu}, \quad k^{\mu} = (p_2 - p_1)^{\mu} \,. \tag{A.41}$$

(The vertices for *transverse* gluons, $\Gamma_{FT\overline{F}}$, clearly contain only the first eight, eqs. (A.38/39), of these matrices). In this basis, the restrictions imposed by C invariance, which leave nine of the twelve scalar functions W_i independent, take rather simple forms. In particular,

$$W_i\left(p_1^2, k^2, p_2^2\right) = W_{i+1}\left(p_2^2, k^2, p_1^2\right) \quad (i = 2 \text{ or } 6)\,, \tag{A.42}$$

$$W_i\left(p_1^2, k^2, p_2^2\right) = W_i\left(p_2^2, k^2, p_1^2\right) \quad (i = 1, 4, 5, 8)\,, \tag{A.43}$$

so that the transverse part has six independent invariant functions, with four of these being symmetric under interchange of the fermion and anti-fermion variables.

Construction of the FDRA sequence is greatly simplified by anticipating that DS self-consistency will dictate, for the fermionic variables, the correlation between pole and matrix structures indicated in (A.35), leaving only "matrix-valued poles", with the ordering of matrices adopted in (A.38 - 40) always observed. The resulting transverse, nonperturbative vertex of zeroth perturbative order, obeying conditions (2.3/2.4), will be written in the form of a partial-fraction decomposition:

$$\left[\Gamma^{\nu}_{FT\overline{F}}(p_1,k,-p_2)\right]^{[r,0]} = t^{\nu\mu}(k)\left\{C^{\mu}_{0,r}(p_1,-p_2) + \sum_{t=1}^{r} C^{\mu}_{t,r}(p_1,-p_2)\left(\frac{\Lambda^2}{k^2 + \overline{u}'_{r,2t}\Lambda^2}\right)\right\}\,, \tag{A.44}$$

where

$$C^{\mu}_{0,r}(p_1,-p_2) = \gamma^{\mu} + \sum_{s=1}^{r} z^{(r)}_{0,1,s}\left[\frac{\Lambda}{\not{p}_1+\kappa'_{r,2s}}\gamma^{\mu} + \gamma^{\mu}\frac{\Lambda}{\not{p}_2+\kappa'_{r,2s}}\right]$$
$$+ \sum_{s,s'=1}^{r} \frac{\Lambda}{\not{p}_1+\kappa'_{r,2s}}\left[z^{(r)}_{0,4,s,s'}\gamma^{\mu} + z^{(r)}_{0,5,s,s'}\left(\frac{r^{\mu}}{\Lambda}\right)\right]\frac{\Lambda}{\not{p}_2+\kappa'_{r,2s'}}\,, \tag{A.45}$$
$$+\left[\text{terms with } (k^2)^n\,,\ 1 \le n \le \tfrac{r-1}{2}\right]$$

$$C^{\mu}_{t,r}(p_1,-p_2) = z^{(r)}_{t,0}\gamma^{\mu} + z^{'(r)}_{t,0}\left(\frac{r^{\mu}}{\Lambda}\right) + z^{''(r)}_{t,0}\left(\frac{\not{p}_1}{\Lambda}\gamma^{\mu} + \gamma^{\mu}\frac{\not{p}_2}{\Lambda}\right) +$$
$$+ \sum_{s=1}^{r}\frac{\Lambda}{\not{p}_1+\kappa'_{r,2s}}\left[\left(z^{(r)}_{t,1,s} + z^{'(r)}_{t,1,s}\frac{p_2^2}{\Lambda^2}\right)\gamma^{\mu} + z^{(r)}_{t,2,s}\gamma^{\mu}\frac{\not{p}_2}{\Lambda} + z^{(r)}_{t,3,s}\frac{r^{\mu}}{\Lambda}\right.$$
$$\left.+z^{'(r)}_{t,3,s}\frac{r^{\mu}\not{p}_2}{\Lambda^2}\right] + \sum_{s=1}^{r}\left[\left(z^{(r)}_{t,1,s} + z^{'(r)}_{t,1,s}\frac{p_1^2}{\Lambda^2}\right)\gamma^{\mu} + z^{(r)}_{t,2,s}\frac{\not{p}_1}{\Lambda}\gamma^{\mu}\right. \tag{A.46}$$
$$\left.+z^{(r)}_{t,3,s}\frac{r^{\mu}}{\Lambda} + z^{'(r)}_{t,3,s}\frac{\not{p}_1 r^{\mu}}{\Lambda^2}\right]\frac{\Lambda}{\not{p}_2+\kappa'_{r,2s}} + \sum_{s,s'=1}^{r}\frac{\Lambda}{\not{p}_1+\kappa'_{r,2s}}\times$$
$$\times\left[z^{(r)}_{t,4,s,s'}\gamma^{\mu} + z^{(r)}_{t,5,s,s'}\frac{r^{\mu}}{\Lambda}\right]\frac{\Lambda}{\not{p}_2+\kappa'_{r,2s'}}\,.$$

Again, preservation of perturbative renormalizability turns out to be a requirement slightly stronger than (2.3/2.4), imposing extra restrictions in the case where p_1, p_2 run to infinity in a loop while k stays outside the loop and fixed. For the vertex to behave no worse than a constant at large loop momenta in this situation requires the primed coefficients in (A.46), or sums of these, to vanish:

$$z^{'(r)}_{t,0} = z^{''(r)}_{t,0} = 0 \qquad (1 \le t \le r,\ \text{all } r)\,; \tag{A.47}$$

$$\sum_{s=1}^{r} z^{'(r)}_{t,1,s} = \sum_{s=1}^{r} z^{'(r)}_{t,3,s} = 0 \qquad (1 \le t \le r,\ \text{all } r)\,. \tag{A.48}$$

In the remaining terms, mass scales have for simplicity been written in a form appropriate for the chiral limit, $\hat{m}_f = 0$. At $\hat{m}_f \neq 0$, those mass scales in (A.45/46) that are not forced by condition (2.3) to be pure multiples of Λ should be generalized in an obvious way, following the pattern of (A.28/29). Formally, this can be accounted for by allowing for a dependence on the dimensionless ratio $\hat{m}_f/\Lambda$ in the relevant dimensionless z coefficients. In fact, in sufficiently high loop orders, *all* nonperturbative coefficients, including those of the gluon-ghost sector, will eventually depend on *all* $\hat{m}_F/\Lambda$ ratios.

Figure Captions

Fig. 1 Distribution of poles (crosses) and zeroes (circles) for nonperturbatively modified propagation functions $D_T^{[r,0]}(k^2)$ in (A) "particle" subsequence, (B) "quasiparticle" subsequence.

Fig. 2 Diagrammatic form of Dyson-Schwinger equation for two-point vertex (negative-inverse propagator) of gluon field, showing one-DS-loop terms (A ... D) and two-DS-loops terms (E&F) of dressing functional.

Fig. 3 One-particle-reducible contributions to four-point amplitudes of elementary QCD fields that seem to produce causality violation in S-matrix elements for the "quasiparticle" subsequence.

Fig. 4 Qualitative behavior of propagation function $\tilde{D}$ in Euclidean coordinate space for free massive propagator (narrow line) and a "quasiparticle" propagator of type $\tilde{D}^{[1,0]}$ (broad line)

Fig. 5 Schematic view of probability flow through state space of (I) an ordinary scattering system and (II) an asymptotically incomplete system with a closed subspace $\mathcal{H}_C$.

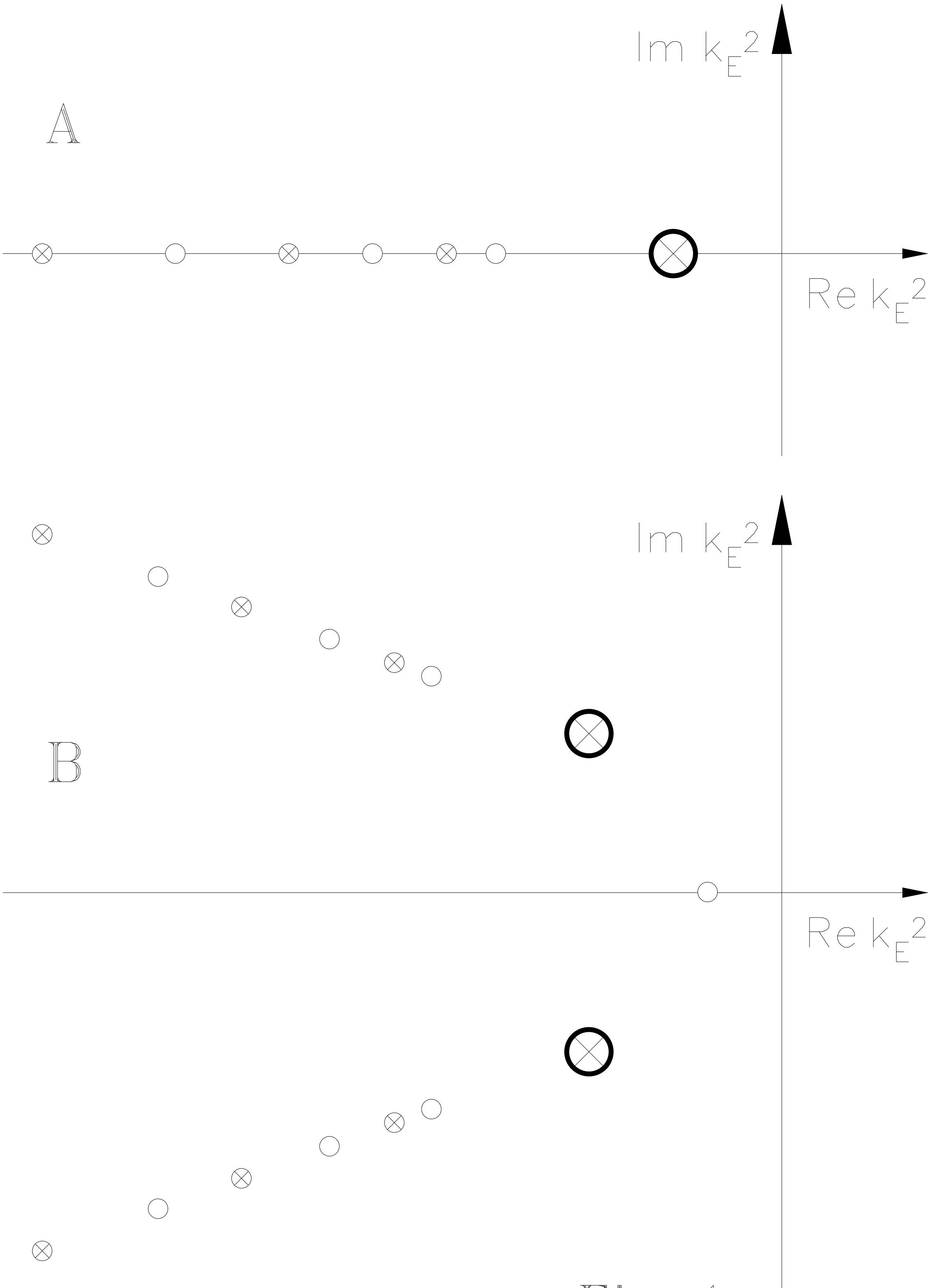

Fig. 1

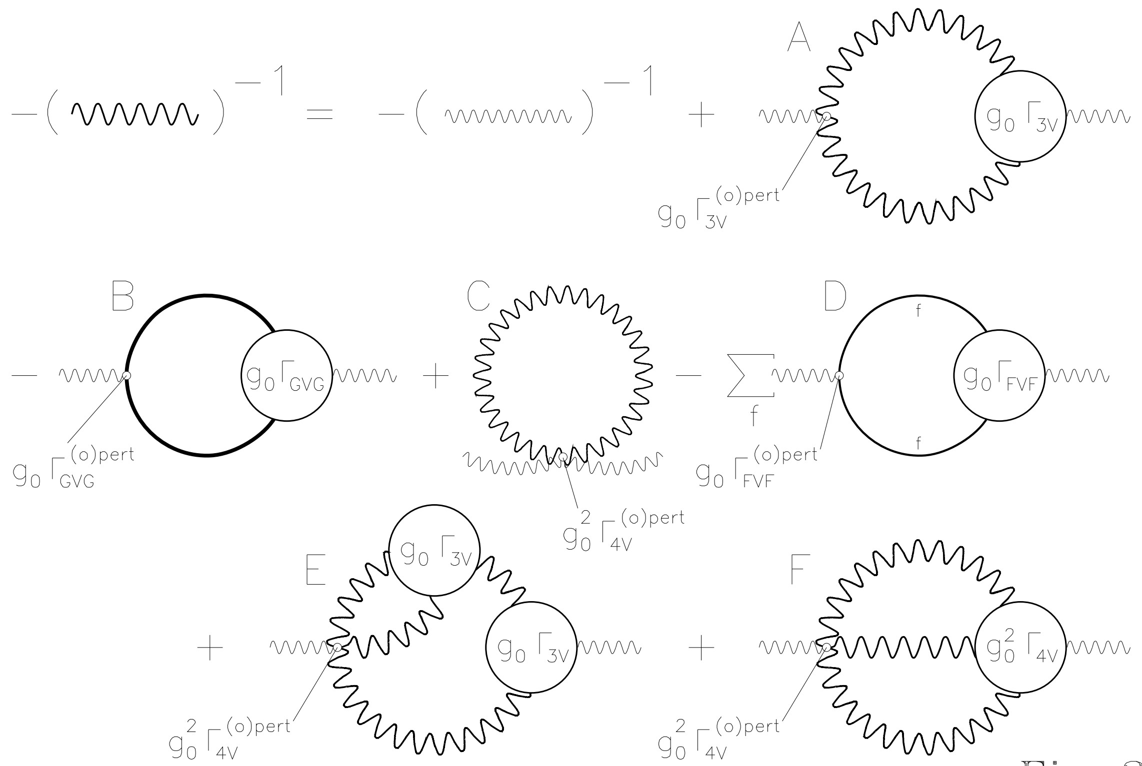
A
B
C
D
E
F
$-(\ \ )^{-1} = -(\ \ )^{-1} +$
$g_0\,\Gamma_{3V}$
$g_0\,\Gamma_{3V}^{(o)pert}$
$g_0\,\Gamma_{GVG}$
$g_0\,\Gamma_{GVG}^{(o)pert}$
$g_0^2\,\Gamma_{4V}^{(o)pert}$
$\sum_f$
f
$g_0\,\Gamma_{FVF}$
$g_0\,\Gamma_{FVF}^{(o)pert}$
$g_0^2\,\Gamma_{4V}$

Fig. 2

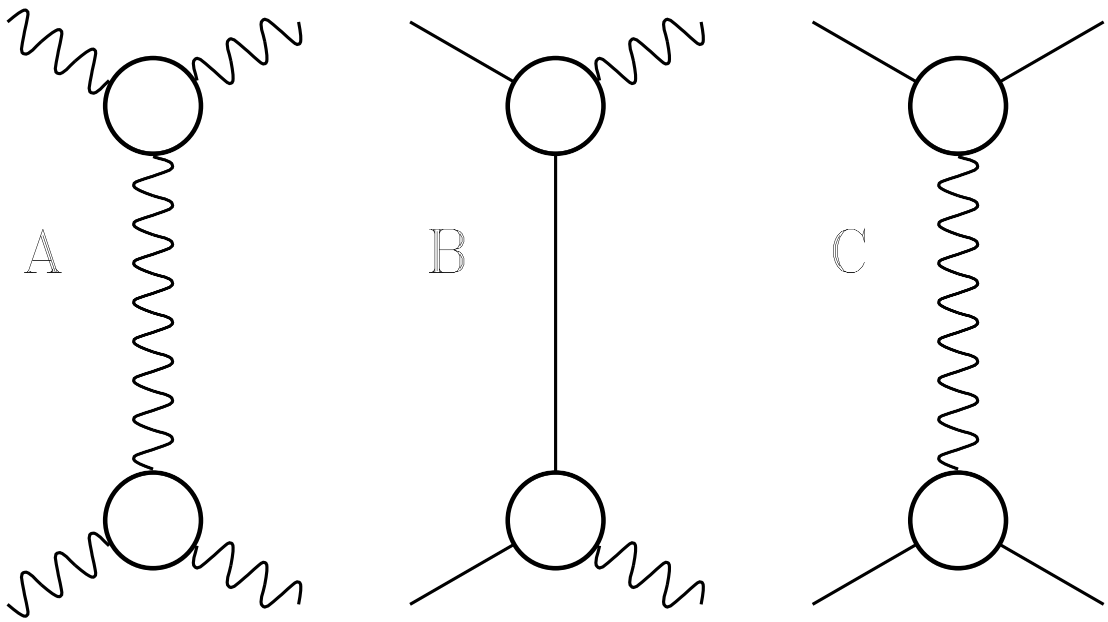

Fig. 3

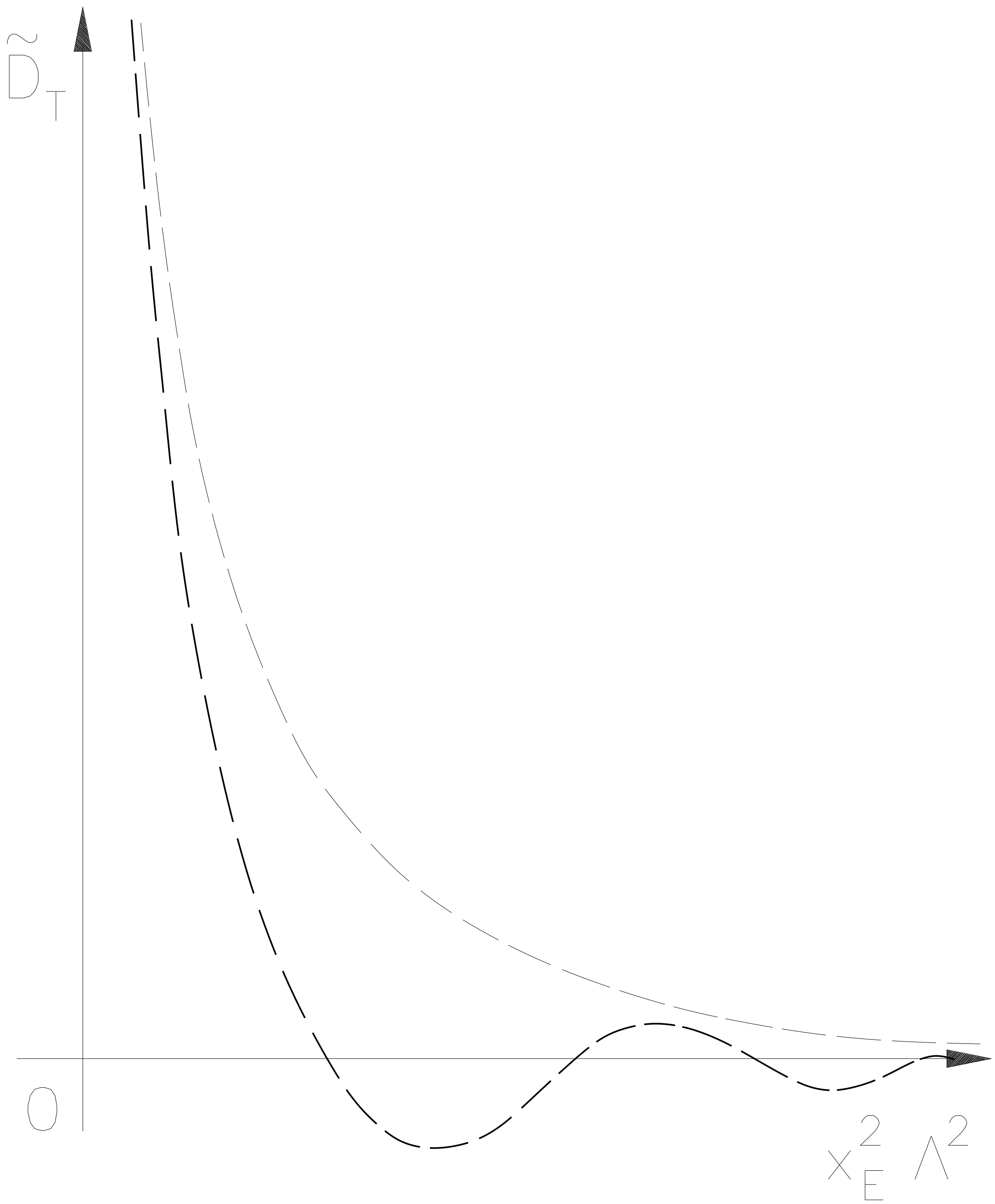

Fig. 4

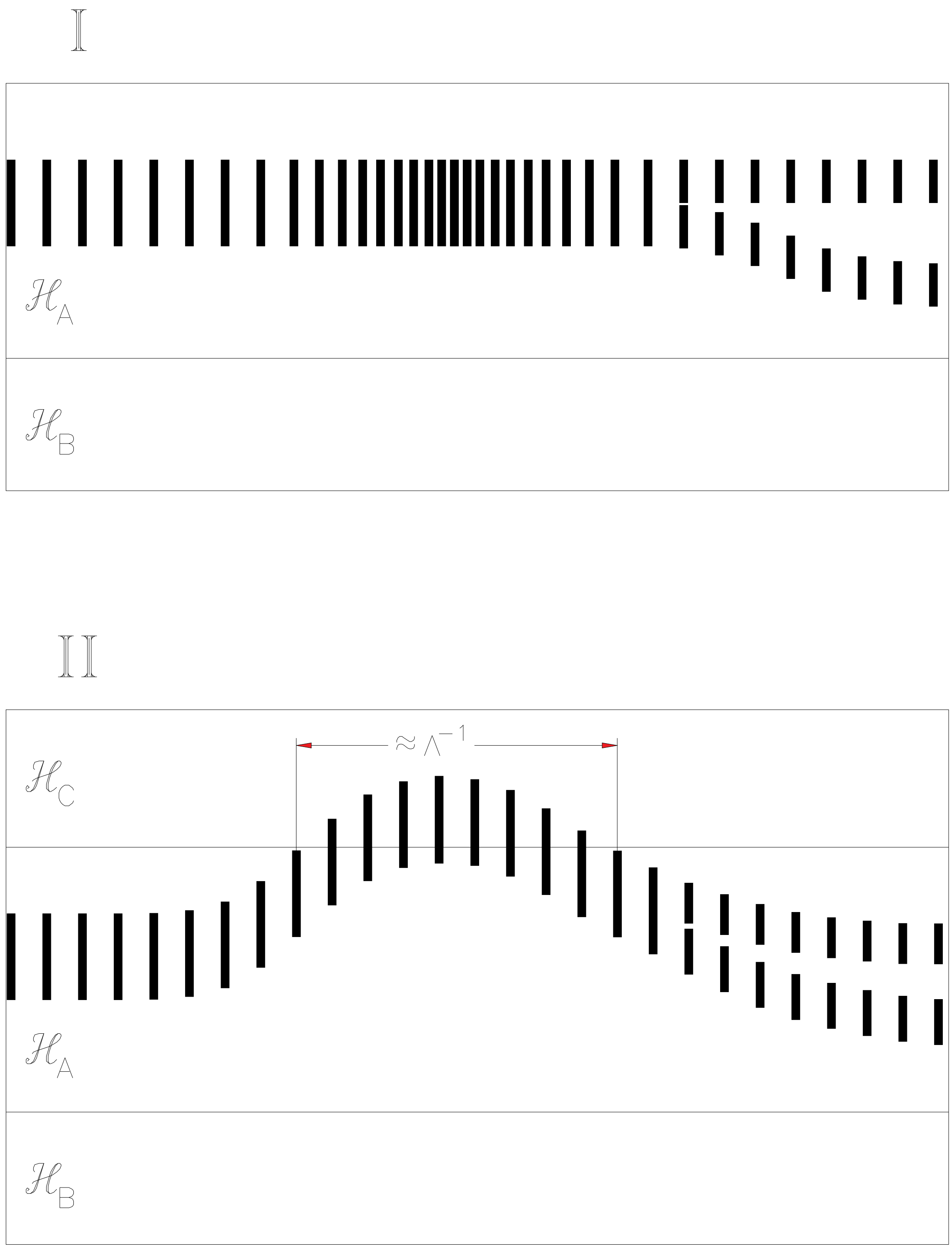
I
$\mathcal{H}_A$
$\mathcal{H}_B$
II
$\approx \Lambda^{-1}$
$\mathcal{H}_C$
$\mathcal{H}_A$
$\mathcal{H}_B$

Fig. 5